Blackboard exam 2, Mathematics Mathematics 3501, Section ETY6
Starts: 6:55 pm, Thurs, Mar 18; ends: 7:45 pm (late submission loses points).
Instructor: Attila Máté

1.*a*) In a factory, parts are manufactured by three machines, M_1, M_2, and M_3 in proportions 10 : 30 : 60. The percentages 4%, 7%, and 3% of these parts are defective, respectively. Find the probability that a randomly chosen part is defective.

Solution. Write A for the event that a part is defective, and M_i with $i = 1, 2, 3$ for the event that it was manufactured on machine i. We have

$$P(A) = \sum_{i=1}^{3} P(A|M_i)P(M_i) = .04 \cdot .1 + .07 \cdot .3 + .03 \cdot .6 = .043.$$

b) Find the probability that a defective part was manufactured on the third machine.

Solution. Using Bayes's theorem, we have

$$P(M_3|A) = \frac{P(A|M_3)P(M_3)}{\sum_{i=1}^{3} P(A|M_i)P(M_i)} = \frac{.03 \cdot .6}{.04 \cdot .1 + .07 \cdot .3 + .03 \cdot .6} = \frac{.018}{.043} \approx .4186;$$

note that the denominator in the third member is the same as the answer to part *a*).

2. Let X be the random variable with density function

$$f_X(x) = \begin{cases} 4x^{-5} & \text{if } x \ge 1, \\ 0 & \text{otherwise.} \end{cases}$$

a) Find the expectation of X.

Solution. We have

$$\mathrm{E}(X) = \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_{1}^{\infty} 4x^{-4}\,dx = \lim_{A\to\infty} \int_{1}^{A} 4x^{-4}\,dx = \lim_{A\to\infty} \left(-\frac{4}{3} \cdot x^{-3}\right)\Big|_{x=1}^{x=A}$$
$$= \lim_{A\to\infty} \left(-\frac{4}{3} \cdot A^{-3} + \frac{4}{3}\right) = \frac{4}{3}.$$

b) Find the variance of X.

Solution. We have $\mathrm{Var}(X) = \mathrm{E}\left(X^2\right) - \left(\mathrm{E}(X)\right)^2$. Here

$$\mathrm{E}\left(X^2\right) = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx = \int_{1}^{\infty} 4x^{-3}\,dx = \lim_{A\to\infty} \int_{1}^{A} 4x^{-4}\,dx = \lim_{A\to\infty} \left(-2 \cdot x^{-2}\right)\Big|_{x=1}^{x=A}$$
$$= \lim_{A\to\infty} \left(-2 \cdot A^{-2} + 2\right) = 2.$$

Hence

$$\mathrm{Var}(X) = \mathrm{E}\left(X^2\right) - \left(\mathrm{E}(X)\right)^2 = 2 - \frac{16}{9} = \frac{2}{9}.$$

c) Find the expectation of X^3.

Solution. We have

$$\mathrm{E}(X^3) = \int_{-\infty}^{\infty} x^3 f_X(x)\,dx = \int_{-\infty}^{\infty} 4x^{-2}\,dx = \lim_{A\to\infty} \int_{1}^{A} 4x^{-2}\,dx = \lim_{A\to\infty} \left(-4 \cdot x^{-1}\right)\Big|_{x=1}^{x=A}$$
$$= \lim_{A\to\infty} \left(-4 \cdot A^{-3} + 4\right) = 4.$$

3. *a*) Let A, B, and C be events, and let I_A, I_B, and I_C be their indicator variables. Describe the event whose probability equals the expectation

$$\mathrm{E}\big(1-(1-I_A)(1-I_B)(1-I_C)\big)$$

Solution. The event is

$$\big(A^* \cap B^* \cap C^*\big)^* = A \cup B \cup C.$$

b) Given events, use indicator variables to derive the formula for the event described in the answer to Part *a*).

Solution. We have

$$\begin{aligned}
\mathrm{P}(A \cup B \cup C) &= \mathrm{E}\big(1-(1-I_A)(1-I_B)(1-I_C)\big) \\
&= \mathrm{E}\big(1-(1-I_A-I_B-I_C+I_AI_B+I_AI_C+I_BI_C-I_AI_BI_C)\big) \\
&= \mathrm{E}(I_A+I_B+I_C-I_AI_B-I_AI_C-I_BI_C+I_AI_BI_C) \\
&= \mathrm{E}(I_A)+\mathrm{E}(I_B)+\mathrm{E}(I_C)-\mathrm{E}(I_AI_B)-\mathrm{E}(I_AI_C)-\mathrm{E}(I_BI_C)+\mathrm{E}(I_AI_BI_C) \\
&= \mathrm{P}(A)+\mathrm{P}(B)+\mathrm{P}(C)-\mathrm{P}(A\cap B)-\mathrm{P}(A\cap C)-\mathrm{P}(B\cap C)+\mathrm{P}(A\cap B\cap C).
\end{aligned}$$