1. a) In a factory, parts are manufactured by three machines, \(M_1, M_2,\) and \(M_3\) in proportions 10 : 30 : 60. The percentages 4\%, 7\%, and 3\% of these parts are defective, respectively. Find the probability that a randomly chosen part is defective.

**Solution.** Write \(A\) for the event that a part is defective, and \(M_i\) with \(i = 1, 2, 3\) for the event that it was manufactured on machine \(i\). We have

\[
P(A) = \sum_{i=1}^{3} P(A|M_i)P(M_i) = .04 \cdot .1 + .07 \cdot .3 + .03 \cdot .6 = .043.
\]

b) Find the probability that a defective part was manufactured on the third machine.

**Solution.** Using Bayes’s theorem, we have

\[
P(M_3|A) = \frac{P(A|M_3)P(M_3)}{\sum_{i=1}^{3} P(A|M_i)P(M_i)} = \frac{.03 \cdot .6}{.04 \cdot .1 + .07 \cdot .3 + .03 \cdot .6} = .018 \approx .4186;
\]

note that the denominator in the third member is the same as the answer to part a).

2. Let \(X\) be the random variable with density function

\[
f_X(x) = \begin{cases} 
4x^{-5} & \text{if } x \geq 1, \\
0 & \text{otherwise}. 
\end{cases}
\]

a) Find the expectation of \(X\).

**Solution.** We have

\[
E(X) = \int_{-\infty}^{\infty} x f_X(x) \, dx = \int_{1}^{\infty} 4x^{-4} \, dx = \lim_{A \to \infty} \int_{1}^{A} 4x^{-4} \, dx = \lim_{A \to \infty} \left( -\frac{4}{3} \cdot x^{-3} \right) \bigg|_{x=1}^{x=A} 
\]

\[
= \lim_{A \to \infty} \left( -\frac{4}{3} \cdot A^{-3} + \frac{4}{3} \right) = \frac{4}{3}.
\]

b) Find the variance of \(X\).

**Solution.** We have \(\text{Var}(X) = E(X^2) - (E(X))^2\). Here

\[
E(X^2) = \int_{-\infty}^{\infty} x^2 f_X(x) \, dx = \int_{1}^{\infty} 4x^{-3} \, dx = \lim_{A \to \infty} \int_{1}^{A} 4x^{-4} \, dx = \lim_{A \to \infty} \left( -2 \cdot x^{-2} \right) \bigg|_{x=1}^{x=A} 
\]

\[
= \lim_{A \to \infty} \left( -2 \cdot A^{-2} + 2 \right) = 2.
\]

Hence

\[
\text{Var}(X) = E(X^2) - (E(X))^2 = 2 - \frac{16}{9} = \frac{2}{9}.
\]

c) Find the expectation of \(X^3\).

**Solution.** We have

\[
E(X^3) = \int_{-\infty}^{\infty} x^3 f_X(x) \, dx = \int_{1}^{\infty} 4x^{-2} \, dx = \lim_{A \to \infty} \int_{1}^{A} 4x^{-2} \, dx = \lim_{A \to \infty} \left( -4 \cdot x^{-1} \right) \bigg|_{x=1}^{x=A} 
\]

\[
= \lim_{A \to \infty} \left( -4 \cdot A^{-3} + 4 \right) = 4.
\]
3. a) Let $A$, $B$, and $C$ be events, and let $I_A$, $I_B$, and $I_C$ be their indicator variables. Describe the event whose probability equals the expectation

$$E(1 - (1 - I_A)(1 - I_B)(1 - I_C))$$

**Solution.** The event is

$$(A \cap B \cap C)^* = A \cup B \cup C.$$  

b) Given events, use indicator variables to derive the formula for the event described in the answer to Part a).

**Solution.** We have

$$P(A \cup B \cup C) = E(1 - (1 - I_A)(1 - I_B)(1 - I_C))$$

$$= E(1 - (1 - I_A - I_B - I_C + I_A I_B + I_A I_C + I_B I_C - I_A I_B I_C))$$

$$= E(I_A + I_B + I_C - I_A I_B - I_A I_C - I_B I_C + I_A I_B I_C)$$

$$= E(I_A) + E(I_B) + E(I_C) - E(I_A I_B) - E(I_A I_C) - E(I_B I_C) + E(I_A I_B I_C)$$

$$= P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C).$$