1. In a series of trials, you toss a pair of coins. What is the probability that on the 6th trial, a pair of heads come of the 4th time.

**Solution.** If $X$ is a random variable with negative binomial distribution with parameters $(4, 1/4)$, the probability is $P(X = 6)$. For a random variable with negative binomial distribution with parameters $(r, p)$, we have

$$P(X = n) = \binom{n-1}{r-1} p^r (1-p)^{n-r}.$$  

Hence the probability in question is

$$P(X = 6) = \binom{6-1}{4-1} \left(\frac{1}{4}\right)^4 (\frac{3}{4})^2 = \frac{5}{2} \left(\frac{1}{4}\right)^4 (\frac{3}{4})^2$$

$$= 10 \cdot \frac{1}{4^4} \cdot \frac{3^2}{4^2} = \frac{90}{2048} = \frac{45}{1024} \approx 0.044021972972656.$$  

2. Find

$$\left(\begin{array}{c} -3 \\ 15 \end{array} \right).$$

Show all your calculations. No credit for only giving the answer.

**Solution.** We have

$$\binom{\alpha}{k} = \frac{1}{k!} \prod_{j=0}^{k-1} (\alpha - j).$$

Hence

$$\left(\begin{array}{c} -3 \\ 15 \end{array} \right) = \frac{1}{15!} \prod_{j=0}^{14} (-3 - j) = \frac{(-3)(-4)(-5) \ldots (-15)(-16)(-17)}{1 \cdot 2 \cdot 3 \ldots \cdot 14 \cdot 15} = -\frac{16 \cdot 17}{2} = 8 \cdot 17 = -136.$$  

3.a) Given the double integral $\iint_D (x-y)e^{x^2-y^2} \, dxdy$, where $D = \{(x, y) : 0 \leq x+y \leq 1 \, \& \, 0 \leq x-y \leq 2\}$, rewrite the double integral with the new variables $u = x+y$ and $v = x-y$, and then rewrite the resulting double integral as an iterated integral.

**Solution.** Writing $u = x+y$ and $v = x-y$, the integrand can be written as

$$(x-y)e^{x^2-y^2} = (x-y)e^{(x+y)(x-y)} = ve^{uv}.$$  

The set corresponding to $D$ is

$$R = \{(x+y, x-y) : 0 \leq x+y \leq 1 \, \& \, 0 \leq x-y \leq 2\} = \{(u, v) : 0 \leq u \leq 1 \, \& \, 0 \leq v \leq 2\}.$$  

We have $x = (u+v)/2$ and $y = (u-v)/2$, and so the Jacobian is

$$\begin{vmatrix} \frac{\partial (x, y)}{\partial (u, v)} \\ \frac{\partial (x, y)}{\partial (u, v)} \end{vmatrix} = \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix} = -\frac{1}{2}.$$  

We are going to use the change of variables formula

$$\iint_D f(x, y) \, dxdy = \iint_R f(x(u, v), y(u, v)) \left| \frac{\partial (x(u, v), y(u, v))}{\partial (u, v)} \right| dudv,$$  

where $f(x(u, v), y(u, v))$ is the integrand in the new variables.
where the vertical bars $|\cdot|$ indicate absolute value, $R = \{(u, v) : (x(u, v), y(u, v)) \in D\}$, and the mapping $(u, v) \mapsto (x(u, v), y(u, v))$ is one-to-one, continuously differentiable,\footnote{I.e., the partial derivatives $\partial x(u, v) \partial u$, $\partial x(u, v) \partial v$, $\partial y(u, v) \partial u$, and $\partial y(u, v) \partial v$ are continuous.} with nonzero Jacobian. We obtain

$$\int\int_D (x - y)e^{x^2 - y^2} \, dx \, dy = \int\int_R ve^{uv} \left| \frac{1}{2} \right| \, du \, dv = \frac{1}{2} \int\int_R ve^{uv} \, du \, dv = \frac{1}{2} \int_0^2 dv \int_0^1 ve^{uv} \, du.$$

\(b\) Evaluate the iterated integral obtained in part \(a\).

\textbf{Solution.} We have

$$\int\int_D (x - y)e^{x^2 - y^2} \, dx \, dy = \frac{1}{2} \int_0^2 dv \int_0^1 ve^{uv} \, du$$

$$= \frac{1}{2} \int_0^2 e^{uv} \bigg|_{u=0}^1 dv = \frac{1}{2} \int_0^2 (e^v - 1) \, dv = \frac{1}{2} \left( e^v - v \right) \bigg|_0^2 = \frac{(e^2 - 2) - 1}{2} = \frac{e^2 - 3}{2}.$$