Blackboard exam 3, Mathematics Mathematics 3501, Section ETY6
Starts: 6:55 pm, Thurs, Apr 8; ends: 7:45 pm (late submission loses points).
Instructor: Attila Máté

1. In a series of trials, you toss a pair of coins. What is the probability that on the 6th trial, a pair of heads come of the 4th time.

Solution. If X is a random variable with negative binomial distribution with parameters $(4, 1/4)$, the probability is $\mathrm{P}(X = 6)$. For a random variable with negative binomial distribution with parameters (r, p), we have

$$\mathrm{P}(X = n) = \binom{n-1}{r-1} p^r (1-p)^{n-r}.$$

Hence the probability in question is

$$\mathrm{P}(X = 6) = \binom{6-1}{4-1} (1/4)^4 (3/4)^{6-4} = \binom{5}{3} (1/4)^4 (3/4)^2 = \binom{5}{2} (1/4)^4 (3/4)^2$$
$$= 10 \cdot \frac{1}{4^4} \cdot \frac{3^2}{4^2} = \frac{90}{2^{12}} = \frac{45}{2^{11}} = \frac{45}{2048} \approx .021,972,656$$

2. Find

$$\binom{-3}{15}.$$

Show all your calculations. No credit for only giving the answer.

Solution. We have

$$\binom{\alpha}{k} = \frac{1}{k!} \prod_{j=0}^{k-1} (\alpha - j).$$

Hence

$$\binom{-3}{15} = \frac{1}{15!} \prod_{j=0}^{14} (-3-j) = \frac{(-3)(-4)(-5)\dots(-15)(-16)(-17)}{1 \cdot 2 \cdot 3 \cdot \ldots \cdot 14 \cdot 15} = -\frac{16 \cdot 17}{2} = 8 \cdot 17 = -136.$$

3. *a*) Given the double integral $\iint_D (x-y)e^{x^2-y^2}\,dxdy$, where $D = \{(x, y) : 0 \le x+y \le 1 \;\&\; 0 \le x-y \le 2\}$, rewrite the double integral with the new variables $u = x+y$ and $v = x-y$, and then rewrite the resulting double integral as an iterated integral.

Solution. Writing $u = x + y$ and $v = x - y$, the integrand can be written as

$$(x-y)e^{x^2-y^2} = (x-y)e^{(x+y)(x-y)} = ve^{uv}.$$

The set corresponding to D is

$$R = \{(x+y, x-y) : 0 \le x+y \le 1 \;\&\; 0 \le x-y \le 2\} = \{(u, v) : 0 \le u \le 1 \;\&\; 0 \le v \le 2\}.$$

We have $x = (u+v)/2$ and $y = (u-v)/2$, and so the Jacobian is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \partial x/\partial u & \partial y/\partial u \\ \partial x/\partial v & \partial y/\partial v \end{vmatrix} = \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix} = -\frac{1}{2}.$$

We are going to use the change of variables formula

$$\iint_D f(x, y)\,dxdy = \iint_R f(x(u, v), y(u, v)) \left| \frac{\partial(x(u, v), y(u, v))}{\partial(u, v)} \right| dudv,$$

where the vertical bars $|\cdot|$ indicate absolute value, $R = \{(u,v) : (x(u,v), y(u,v)) \in D\}$, and the mapping $(u,v) \mapsto (x(u,v), y(u,v)$ is one-to-one, continuously differentiable,[1] with nonzero Jacobian. We obtain

$$\iint_D (x-y)e^{x^2-y^2}\,dxdy = \iint_R ve^{uv}\left|-\frac{1}{2}\right|\,dudv = \frac{1}{2}\iint_R ve^{uv}\,dudv = \frac{1}{2}\int_0^2 dv\int_0^1 ve^{uv}\,du.$$

b) Evaluate the iterated integral obtained in part *a*).

Solution. We have

$$\begin{aligned}
\iint_D (x-y)e^{x^2-y^2}\,dxdy &= \frac{1}{2}\int_0^2 dv\int_0^1 ve^{uv}\,du \\
&= \frac{1}{2}\int_0^2 e^{uv}\Big|_{u=0}^{1}\,dv = \frac{1}{2}\int_0^2 (e^v-1)\,dv = \frac{1}{2}(e^v-v)\Big|_0^2 = \frac{(e^2-2)-1}{2} = \frac{e^2-3}{2}.
\end{aligned}$$

[1]I.e., the partial derivatives $\partial x(u,v)\partial u$, $\partial x(u,v)\partial v$, $\partial y(u,v)\partial u$, and $\partial y(u,v)\partial v$ are continuous.