1. The numbers
81.1, 84.7, 77.8, 76.4
represent the temperature measured at a certain location and at a certain time. This is regarded as a random
sample from a distribution with mean \( m \) and variance \( \sigma^2 \).

\( \text{(a)} \) Find an unbiased estimate for \( m \).

\textbf{Solution.} Given a random sample \( x_1, x_2, \ldots, x_n \) (i.e., observations of the identically distributed independent
random variables \( X_1, X_2, \ldots, X_n \)), an unbiased estimate for the mean is
\[
\bar{x} \overset{\text{def}}{=} \frac{1}{n} \sum_{k=1}^{n} x_k .
\]
In the present case, \( n = 4 \) and \( x_1 = 81.1, x_2 = 84.7, x_3 = 77.8, \) and \( x_4 = 76.4 \). Hence \( \bar{x} = 80.0 \) is an unbiased
estimate for \( m \).

\( \text{(b)} \) Find an unbiased estimate for \( \sigma^2 \) if it is known that \( m = 79.0 \).

\textbf{Solution.} We have \( \sigma^2 = \text{Var}(X_k) = \text{E}((X_k - m)^2) \) (for all \( k \) with \( 1 \leq k \leq n \), since the variables \( X_k \) are
identically distributed). Given that \( m \) is known, an unbiased estimate for \( \sigma^2 \) is the same as an unbiased
estimate for the mean (expectation) \( \text{E}((X_k - m)^2) \). That is, the estimate for \( \sigma^2 \) we are looking for is
\[
\frac{1}{n} \sum_{k=1}^{n} (x_k - m)^2 .
\]
With the present data, \( m = 79.0 \) and \( x_k \) and \( n \) as above, we have 11.275 as the estimate for \( \sigma^2 \) we are
looking for.

\( \text{(c)} \) Find an unbiased estimate for \( \sigma^2 \) if \( m \) is unknown.

\textbf{Solution.} If \( m \) is not known, an unbiased estimate for \( \sigma^2 \) is
\[
s^2 = \frac{1}{n-1} \sum_{k=1}^{n} (x_k - \bar{x})^2 ,
\]
where \( \bar{x} \) is as above. In the present case, this gives \( s^2 \approx 13.70 \).

2. Let \( X \) and \( Y \) be random variables such that \( V(X) = V(Y) \). Show that
\[
\text{Cov}(X + Y, X - Y) = 0 .
\]

\textbf{Solution.} We have
\[
\text{Cov}(X + Y, X - Y) = \text{Cov}(X, X) + \text{Cov}(X, Y) - \text{Cov}(Y, X) - \text{Cov}(Y, Y) = V(X) - V(Y) = 0 ;
\]
here the second equation holds since \( \text{Cov}(X, Y) = \text{Cov}(Y, X) \), \( \text{Cov}(X, X) = V(X) \), and \( \text{Cov}(Y, Y) = V(Y) \),
and the third equation holds since we assumed that \( V(X) = V(Y) \).

3. Let \( X \) be a random variable that is uniformly distributed on the interval \([a, b]\). Find its moment
generating function.

\textbf{Solution.} We have
\[
f_X(x) = \begin{cases} 
1/(b - a) & \text{if } x \in [a, b], \\
0 & \text{otherwise}, 
\end{cases}
\]
Assuming $t \neq 0$, we have

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f_X(x) \, dx = \frac{1}{b-a} \int_{a}^{b} e^t x \, dx = \frac{e^{tb} - e^{ta}}{(b-a)t}$$

if $t \neq 0$. For $t = 0$ we have

$$M_X(0) = \int_{-\infty}^{\infty} e^{0x} f_X(x) \, dx = \frac{1}{b-a} \int_{a}^{b} dx = 1.$$

Note that it is easy to show by l’Hospital’s rule that $\lim_{t \to 0} M_X(t) = 1$; that is, $M_X(t)$ is continuous everywhere.