Blackboard exam 4, Mathematics Mathematics 3501, Section ETY6
Starts: 6:55 pm, Thurs, Apr 22; ends: 7:45 pm (late submission loses points).
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1. The numbers

$$81.1, \qquad 84.7, \qquad 77.8, \qquad 76.4$$

represent the temperature measured at a certain location and at a certain time. This is regarded as a random sample from a distribution with mean m and variance σ^2.

a) Find an unbiased estimate for m.

Solution. Given a random sample $x_1, x_2, \ldots, x_n$ (i.e., observations of the identically distributed independent random variables $X_1, X_2, \ldots, X_n$), an unbiased estimate for the mean is

$$\bar{x} \overset{def}{=} \frac{1}{n}\sum_{k=1}^{n} x_n.$$

In the present case, $n = 4$ and $x_1 = 81.1$, $x_2 = 84.7$, $x_3 = 77.8$, and $x_4 = 76.4$. Hence $\bar{x} = 80.0$ is an unbiased estimate for m.

b) Find an unbiased estimate for σ^2 if it is known that $m = 79.0$.

Solution. We have $\sigma^2 = \mathrm{Var}(X_k) = \mathrm{E}\big((X_k - m)^2\big)$ (for all k with $1 \le k \le n$, since the variables X_k are identically distributed). Given that m is known, an unbiased estimate for σ^2 us the same as an unbiased estimate for the mean (expectation) $\mathrm{E}\big((X_k - m)^2\big)$. That is, the estimate for σ^2 we are looking for is

$$\frac{1}{n}\sum_{k=1}^{n}(x_k - m)^2.$$

With the present data, $m = 79.0$ and x_k and n as above, we have 11.275 as the estimate for σ^2 we are looking for.

c) Find an unbiased estimate for σ^2 if m is unknown.

Solution. If m is not known, an unbiased estimate for σ^2 is

$$s^2 = \frac{1}{n-1}\sum_{k=1}^{n}(x_k - \bar{x})^2,$$

where $\bar{x}$ is as above. In the present case, this gives $s^2 \approx 13.70$.

2. Let X and Y be random variables such that $\mathrm{V}(X) = \mathrm{V}(Y)$. Show that

$$\mathrm{Cov}(X + Y, X - Y) = 0.$$

Solution. We have

$$\mathrm{Cov}(X+Y, X-Y) = \mathrm{Cov}(X,X) + \mathrm{Cov}(X,Y) - \mathrm{Cov}(Y,X) - \mathrm{Cov}(Y,Y) = \mathrm{V}(X) - \mathrm{V}(Y) = 0;$$

here the second equation holds since $\mathrm{Cov}(X,Y) = \mathrm{Cov}(Y,X)$, $\mathrm{Cov}(X,X) = \mathrm{V}(X)$, and $\mathrm{Cov}(Y,Y) = \mathrm{V}(Y)$, and the third equation holds since we assumed that $V(X) = \mathrm{V}(Y)$.

3. Let X be a random variable that is uniformly distributed on the interval $[a, b]$. Find its moment generating function.

Solution. We have

$$f_X(x) = \begin{cases} 1/(b-a) & \text{if } x \in [a,b], \\ 0 & \text{otherwise,} \end{cases}$$

Assuming $t \neq 0$, we have

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f_X(x)\, dx = \frac{1}{b-a} \int_a^b e^t x\, dx = \frac{e^{tb} - e^{ta}}{(b-a)t}$$

if $t \neq 0$. For $t = 0$ we have

$$M_X(0) = \int_{-\infty}^{\infty} e^{0x} f_X(x)\, dx = \frac{1}{b-a} \int_a^b \, dx = 1.$$

Note that it is easy to show by l'Hospital's rule that $\lim_{t \to 0} M_X(t) = 1$; that is, $M_X(t)$ is continuous everywhere.