Blackboard exam 5, Mathematics Mathematics 2501, Section TR2
Starts: 2:15 pm, Thurs, May 14; ends: 3:30 pm.
Instructor: Attila Máté

1. Let X be the random variable with density function

$$f_X(x) = \begin{cases} 4x^3 & \text{if } 0 \le x \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

a) Find the expectation of X.

Solution. We have

$$\mathrm{E}(X) = \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_0^1 4x^4\,dx = \frac{4}{5}\cdot x^5\Big|_{x=0}^{x=1} = \frac{4}{5}.$$

b) Find the variance of X.

Solution. We have $\mathrm{Var}(X) = \mathrm{E}\left(X^2\right) - \left(\mathrm{E}(X)\right)^2$. Here

$$\mathrm{E}\left(X^2\right) = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx = \int_0^1 4x^5\,dx = \frac{4}{6}\cdot x^6\Big|_{x=0}^{x=1} = \frac{2}{3}\cdot x^6\Big|_{x=0}^{x=1} = \frac{2}{3}.$$

Hence

$$\mathrm{Var}(X) = \mathrm{E}\left(X^2\right) - \left(\mathrm{E}(X)\right)^2 = \frac{2}{3} - \frac{16}{25} = \frac{50}{75} - \frac{48}{75} = \frac{2}{75} = .02666\ldots$$

(a repeating decimal).

3. A pair of fair coins are tossed 1200 times. Let X denote the number of times both coins come up head.

a) Write a formula for the probability $P(X = k)$ $(0 \le k \le 1200)$.

Solution. The probability that both coins come up head in any single toss is 1/4. Therefore X has a binomial distribution $X \sim \mathrm{Bin}(1200, 1/4)$. That is,

$$P(X = k) = \binom{1200}{k}\left(\frac{1}{4}\right)^k\left(\frac{3}{4}\right)^{1200-k}.$$

b) Write the sum for the exact probability that $290 \le X \le 310$.

Solution. We have

$$\mathrm{P}(290 \le X \le 310) = \sum_{k=290}^{310} \mathrm{P}(X = k) = \sum_{k=290}^{310} \binom{1200}{k}\left(\frac{1}{4}\right)^k\left(\frac{3}{4}\right)^{1200-k}.$$

c) Use a well-known approximation to calculate the approximate probability that $290 \le X \le 310$.

Solution. We use normal approximation with continuity correction. A $\mathrm{Bin}(n, p)$ distribution has expectation np and standard deviation $\sqrt{np(1-p)}$. Accordingly, writing $\mathrm{D}(X)$ for the standard deviation of X, we have

$$\mathrm{E}(X) = 1200\cdot\frac{1}{4} = 300 \quad \text{and} \quad \mathrm{D}(X) = \sqrt{1200\cdot\frac{1}{4}\cdot\frac{3}{4}} = 15.$$

Thus, the distribution of X can be approximated by a that of a normal variable $Y \sim \mathcal{N}(300, 15^2)$, i.e., by a normal variable Y having expectation 300 and standard deviation 15. Writing

$$Z = \frac{Y - 300}{15},$$

Y is a standard normal variable. Hence, using continuity correction,

$$\begin{aligned}
\mathrm{P}(290 \le X \le 310) &\approx \mathrm{P}(289.5 \le Y \le 310.5) \\
&= \mathrm{P}\left(\frac{289.5 - 300}{15} \le \frac{Y - 300}{15} \le \frac{310.5 - 300}{15}\right) = \mathrm{P}\left(-.7 \le \frac{Y - 300}{15} \le .7\right) \\
&= \mathrm{P}\left(-.7 \le Z \le .7\right) = \mathrm{P}(Z \le .7) - P(Z < -.7) = \mathrm{P}(Z \le .7) - \mathrm{P}(Z > .7) \\
&= \mathrm{P}(Z \le .7) - \big(1 - \mathrm{P}(Z \le .7)\big) = 2\,\mathrm{P}(Z \le .7) - 1 = 2\Phi(.7) - 1;
\end{aligned}$$

here, the Φ denotes the distribution function of the standard normal distribution. The first equality in the last line follows since the distribution of Z is symmetric about 0. We can use the normal distribution table to find that $\Phi(.7) \approx .7580$. Thus, we gave

$$\mathrm{P}(290 \le X \le 310) \approx .5160.$$

Calculating the sum Part *b*) directly, with a short program written in the programming language Maxima, we get a more precise answer

$$\mathrm{P}(290 \le X \le 310) \approx .5160583906382258,$$

showing that the result obtained by the normal approximation is almost correct to four decimal places; not quite, since rounding the more precise result to four decimal places, we obtain

$$\mathrm{P}(290 \le X \le 310) \approx .5161.$$