The natural exponential function

Attila Máté
Brooklyn College of the City University of New York

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1 The natural exponential function for real x

1.1 Bernoulli's inequality

For every real $x \ge -1$ and every integer $n \ge 1$ we have

$$(1+x)^n \ge 1 + nx. \tag{1}$$

Equality here holds only in case $n = 1$ or $x = 0$.[1] This is easily proved by induction on n. Indeed, (1) is true for $n = 1$, because then it states that $1 + x \le 1 + x$. Let $n \ge 1$, and assume (1) is true for this n. Then we have

$$(1+x)^{n+1} = (1+x)^n(1+x) \ge (1+nx)(1+x) = 1 + (n+1)x + nx^2 \ge 1 + (n+1)x.$$

[1] We require $n \ge 1$ since the case $n = 0$ and $x = -1$ leads to the meaningless expression 0^0. Clearly, the inequality is also true if $n = 0$ and $x > -1$.

Note that the first inequality here results by multiplying both sides of equation (1) by $1+x$. This is allowed, since $1+x$ is non-negative (multiplying an inequality by a negative number reverses the inequality). This completes the proof of (1).

Note that in the last displayed formula, strict inequality holds instead of the last inequality unless $x=0$. This also establishes our comment about equality in (1).

1.2 The function $\exp(x)$

We write

$$\exp(x) = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n. \tag{2}$$

To establish the existence of the limit on the right, we first assume that $x \ge 0$. By the binomial theorem, we have

$$\left(1+\frac{x}{n}\right)^n = \sum_{k=0}^{n}\binom{n}{k}\frac{x^k}{n^k} = \sum_{k=0}^{n}\frac{1}{k!}x^k\prod_{j=1}^{k-1}\left(1-\frac{j}{n}\right) \le \sum_{k=0}^{\infty}\frac{x^k}{k!}.$$

To see that the series on the right is convergent, one may observe that for m, n, and x with $0 \le x < m < n$ we have

$$\sum_{k=m}^{n}\frac{x^k}{k!} \le x^m\sum_{k=m}^{\infty}\left(\frac{x}{m}\right)^{k-m}, \tag{3}$$

and the geometric series on the right is convergent. We can also conclude from the above equations that

$$\left(1+\frac{x}{n}\right)^n \ge \sum_{k=0}^{m}\frac{1}{k!}x^k\prod_{j=1}^{k-1}\left(1-\frac{j}{n}\right)$$

for any positive $m \le n$. Keeping m fixed while making $n\to\infty$, it follows that

$$\liminf_{n\to\infty}\left(1+\frac{x}{n}\right)^n \ge \sum_{k=0}^{m}\frac{x^k}{k!} \tag{4}$$

for any $x \ge 0$. Since m can be any positive integer here, it follows that

$$\exp(x) = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = \sum_{k=0}^{\infty}\frac{x^k}{k!} \tag{5}$$

for $x \ge 0$.[2] In particular, the limit in (2) exists for $x \ge 0$. From this equation it follows that

$$1+x \le \exp(x) \le \sum_{k=0}^{\infty}x^k = \frac{1}{1-x} \tag{6}$$

for x with $0 \le x \le 1$. Thus it is easy to show that

$$\lim_{x\searrow 0}\frac{\exp(x)-1}{x} = 1. \tag{7}$$

[2]This equation will be established for all x in Subsection 2.1.

1.3 The identity $\exp(x)\exp(-x) = 1$

For x and n with $0 \le x < n$ we have

$$1 \ge \left(1+\frac{x}{n}\right)^n \left(1-\frac{x}{n}\right)^n = \left(1-\frac{x^2}{n^2}\right)^n \ge 1-\frac{x^2}{n},$$

where the second inequality holds in view of Bernoulli's inequality (1). Making $n \to \infty$, the existence of the limit in (2) follows also for negative x, and we obtain

$$\exp(x)\exp(-x) = 1. \tag{8}$$

This equation and equation (7) allow us to conclude that

$$\lim_{x\to 0}\frac{\exp(x)-1}{x} = 1. \tag{9}$$

1.4 The identity $\exp(x)\exp(y) = \exp(x+y)$

Assume x and y are non-negative and $n > \sqrt{xy}$. We have

$$\begin{aligned}\left(1+\frac{x+y}{n}\right)^n &\le \left(1+\frac{x}{n}\right)^n \left(1+\frac{y}{n}\right)^n = \left(1+\frac{x+y}{n}+\frac{xy}{n^2}\right)^n \\ &\le \left(1+\frac{x+y}{n}\right)^n \left(1+\frac{xy}{n^2}\right)^n \le \left(1+\frac{x+y}{n}\right)^n \Big/ \left(1-\frac{xy}{n^2}\right)^n \\ &\le \left(1+\frac{x+y}{n}\right)^n \Big/ \left(1-\frac{xy}{n}\right),\end{aligned}$$

where the last inequality follows from Bernoulli's inequality (1). Making $n \to \infty$, we obtain

$$\exp(x+y) = \exp(x)\exp(y) \tag{10}$$

for non-negative x and y. Using equation (8), this equation can also be established for all real x and y. With the aid of this identity and equation (9), we can easily show that

$$\frac{d}{dx}\exp(x) = \exp(x). \tag{11}$$

Hence it also follows that the function $\exp(x)$ is continuous.

1.5 The number e and the function e^x

It is easy to show that $\big(\exp(x)\big)^r = \exp(rx)$ for any real x and rational r. All that is involved in showing this is changing the order of taking a limit and applying a continuous function (namely, raising to a rational power), and noting that a convergent sequence and any of its subsequences have the same limit.[3] Defining the number e as $\exp(1)$, we then have $e^r = \exp(r)$ for any rational r. As for irrational x, there is no reasonable way to define e^x other than approximating x with rationals. As $\exp(x)$ is continuous, it can be seen that such a definition is sound,[4] and it follows that $e^x = \exp(x)$ for all real x with such a definition.

[3] Alternatively, one can also use (9) to show this equation.

[4] Soundness means that if r_n are rationals such that $\lim_{n\to\infty} r_n = x$ then the limit $\lim_{n\to\infty}\exp(r_n)$ exists, and if one takes another sequence of rationals s_n with $\lim_{n\to\infty} s_n = x$ then $\lim_{n\to\infty}\exp(r_n) = \lim_{n\to\infty}\exp(s_n)$.

2 The natural exponential function for complex x

2.1 Extending the approach to complex numbers

The above discussion can be extended to complex numbers by using somewhat more about infinite series. One can establish equation (5) for all complex x by pointing out that (3) with $|x|$ replacing x implies the absolute convergence of the series on the right-hand side of (5). Then inequality (4) need to be replaced with

$$\limsup_{n\to\infty}\left|\left(1+\frac{x}{n}\right)^n-\sum_{k=0}^{m}\frac{x^k}{k!}\right|\le\sum_{k=m+1}^{\infty}\frac{|x|^k}{k!}.$$

The use of lim sup here and of lim inf in (4) can be avoided at the price of some minor additional complications. Having established (5) for all complex x, the existence of the limit in (2) now follows and we have

$$|\exp(x)-1|\le\frac{|x|}{1-|x|}\qquad\text{for}\quad|x|<1.$$

Instead of using (5) to establish this inequality, we may also point out that it directly follows from Lemma 1 below. This inequality will replace the estimates in (6). While the rest of the discussion could be continued with making use of various properties of power series, we will avoid any further use of expansion (5) in these notes. In particular, we will show how Euler's formula (15) can be established without the use of power series.

The last inequality can be used to prove (9) directly. We will show how to establish (10) for all complex x and y next in Subsection 2.2. Thus, equations (5), (9), (10), and (11) can be extended to all complex numbers x and y.

2.2 The identity $\exp(x)\exp(y)=\exp(x+y)$ for complex x and y.

To extend (10) to complex x and y, we can use the following

Lemma 1. *Let $n>0$ be an integer and z be a complex number such that $n|z|<1$. Then*

$$\left|(1+z)^n-1\right|\le\frac{n|z|}{1-n|z|}.$$

Proof. We have

$$\begin{aligned}\left|(1+z)^n-1\right|&=\left|z\sum_{k=0}^{n-1}(1+z)^k\right|\le|z|\sum_{k=0}^{n-1}(1+|z|)^k\\&\le|z|\sum_{k=0}^{n-1}\frac{1}{(1-|z|)^k}\le|z|\sum_{k=0}^{n-1}\frac{1}{1-k|z|}\le|z|\sum_{k=0}^{n-1}\frac{1}{1-n|z|}=\frac{n|z|}{1-n|z|}\end{aligned}$$

where the penultimate[5] inequality follows from Bernoulli's inequality (1).[6] □

As an immediate consequence, we have

[5] The one before the last.

[6] The case $k=0$ is obvious without Bernoulli's inequality. Note that in Bernouilli's inequality we required $n\ge1$, so that would not apply in this case.

Corollary 1. *For $n \geq 1$ let z_n be complex numbers such that $\lim_{n\to\infty} nz_n = 0$. Then*

$$\lim_{n\to\infty}(1+z_n)^n = 1.$$

Let x and y be arbitrary complex numbers. Then

$$\left(1+\frac{x}{n}\right)^n\left(1+\frac{y}{n}\right)^n = \left(1+\frac{x+y}{n}+\frac{xy}{n^2}\right)^n = \left(1+\frac{x+y}{n}\right)^n(1+z_n)^n \tag{12}$$

for large enough n, where

$$z_n = \left(1+\frac{x+y}{n}+\frac{xy}{n^2}\right)\bigg/\left(1+\frac{x+y}{n}\right)-1;$$

making sure that n is large enough guarantees that the denominator here is not 0. It is easy to see that $\lim_{n\to\infty} nz_n = 0$, and so we have $\lim_{n\to\infty}(1+z_n)^n = 1$ by Corollary 1. Making $n \to \infty$, equation (10) follows.

2.3 De Moivre's formula

The complex number $a+bi$, where a and b are real numbers, can be represented in the coordinate plane as the point (a,b). In this case, the x-axis is called the real axis, and the y-axis, the imaginary axis. If this same point is represented as the pair (ρ,θ) in polar coordinates, where $\rho \geq 0$ and θ is real, then ρ is called the absolute value $|a+bi|$ and θ, the argument $\arg(a+bi)$ of $a+bi$; clearly, the argument is only determined up to an integral multiple multiple of 2π.[7] That is, $\arg(a+bi) = \theta_0+2k\pi$ for some θ_0, where k can be arbitrarily chosen as any integer.

If $\rho = |a+bi|$ and $\theta = \arg(a+bi)$, then, clearly

$$a+bi = \rho(\cos\theta + i\sin\theta).$$

The right-hand side is called the trigonometric form of the complex number on the left.

When multiplying two complex numbers, their absolute values get multiplied and their arguments get added. That is, for real ρ_1, ρ_2, θ_1, and θ_2, we have

$$\rho_1(\cos\theta_1 + i\sin\theta_1)\cdot\rho_2(\cos\theta_2 + i\sin\theta_2) = \rho_1\rho_2(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)).$$

This equation easily follows from the addition formulas for sin and cos. A better approach is to verify it directly, using coordinate transformations, and then use this equation to prove the addition formulas for sin and cos. The last equation implies

$$(\cos x + i\sin x)^n = \cos nx + i\sin nx \tag{13}$$

for any integer n and any real x. This is called de Moivre's formula.

2.4 Euler's formula

Observe that

$$\lim_{t\to 0}\frac{t-\sin t}{t} = \lim_{t\to 0}\frac{1-\cos t}{t} = 0.$$

[7] The word "integral" does not have anything to do with integration; is it the adjectival form of the word "integer" (which can be positive, negative, or 0).

Thus, for any real x we have

$$\lim_{n\to\infty} n\left(\left(1+i\frac{x}{n}\right)-\left(\cos\frac{x}{n}+i\sin\frac{x}{n}\right)\right)=0.$$

Hence, for a fixed real x, writing

$$z_n=\frac{1+i\frac{x}{n}}{\cos\frac{x}{n}+i\sin\frac{x}{n}}-1,$$

and noting that

$$\left|\cos\frac{x}{n}+i\sin\frac{x}{n}\right|=1,$$

we obtain that $\lim_{n\to\infty} nz_n=0$. Therefore, by Corollary 1 it follows that

$$\lim_{n\to\infty}(1+z_n)^n=1. \tag{14}$$

We have

$$\left(1+i\frac{x}{n}\right)^n=\left(\cos\frac{x}{n}+i\sin\frac{x}{n}\right)^n(1+z_n)^n=(\cos x+i\sin x)(1+z_n)^n,$$

where the second equation follows from de Moivre's formula (13). Making $n\to\infty$, we obtain

$$\exp(ix)=\lim_{n\to\infty}\left(1+i\frac{x}{n}\right)^n=\cos x+i\sin x \tag{15}$$

for every real x by (14). This is Euler's formula.

3 Comments

3.1 Monotonicity of $\left(1+\frac{x}{n}\right)^n$ for real x

The above approach can be modified to avoid any reference to the series in (5); it it, however, not clear whether these modifications result in essential simplification. To prove the existence of the limit in (2) for real x, we may start with noting that the sequence on the right-hand side is increasing if x is real. More precisely,

$$\left(1+\frac{x}{n}\right)^n\le\left(1+\frac{x}{n+1}\right)^{n+1}\qquad\text{if}\quad n>\max\{-x,0\}; \tag{16}$$

equality here holds only in case $x=0$. Indeed, noting that

$$\left(1+\frac{x}{n}\right)^{-1}=\frac{n}{n+x}=1-\frac{x}{n+x},$$

this inequality is equivalent to

$$1-\frac{x}{n+x}\le\left(\left(1+\frac{x}{n+1}\right)\left(1-\frac{x}{n+x}\right)\right)^{n+1};$$

we used the assumption $n+x>0$ to make sure that in obtaining this inequality, we multiplied the starting inequality by a positive number. The right-hand side here equals

$$\left(1-\frac{x}{(n+x)(n+1)}\right)^{n+1}\ge 1-\frac{x}{n+x},$$

where the inequality holds in view of Bernoulli's inequality (1); equality holds only in case $x=0$ (cf. the comment on equality in Bernoulli's inequality).

3.2 Boundedness of $\left(1+\frac{x}{n}\right)^n$ for complex x

The sequence on the right-hand side of (2) is bounded for all complex x. Indeed, if $|x|<1$, then

$$\left|\left(1+\frac{x}{n}\right)^n\right| \le \left(1+\frac{|x|}{n}\right)^n \le 1 \Big/ \left(1-\frac{|x|}{n}\right)^n \le \frac{1}{1-|x|},$$

where the last inequality holds for holds in view of Bernoulli's inequality (1). If $|x| \ge 1$, let $m>0$ be an integer such that $|x|/m<1$. Then, given any integer $n \ge 1$, let k be an integer with $km>n$. Using the monotonicity of $(1+x/n)^n$ expressed by (16), and then using the last displayed inequality with $|x|/m$ replacing x, we obtain

$$\left|\left(1+\frac{x}{n}\right)^n\right| \le \left(1+\frac{|x|}{n}\right)^n \le \left(1+\frac{|x|}{km}\right)^{km} = \left(1+\frac{|x|/m}{k}\right)^{km} \le \left(\frac{1}{1-|x|/m}\right)^m.$$

Hence, for real x, the existence of the limit in (2) follows from the boundedness and the monotonicity of the sequence on the right-hand side of that formula.

3.3 Convergence of $\left(1+\frac{x}{n}\right)^n$ for complex z

Let $z=x+iy$ be a complex number, where x and y are real. Using equation (12) with iy replacing y, and modifying the definition of z_n accordingly, we can show that

$$\lim_{n\to\infty}\left(1+\frac{x+iy}{n}\right)^n = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n \cdot \lim_{n\to\infty}\left(1+\frac{iy}{n}\right)^n,$$

where we interpret this equation in the sense that if the limits on the right-hand side exist, then so does the limit on the left-hand side, and the two sides are equal. As for the first limit on the right-hand side, its existence can be justified by what we said above in this section, i.e., that we are taking the limit of a bounded increasing sequence. The existence of the second limit can be justified by noting that when we established (15) above, we proved the existence of the limit directly, and we did not rely on the fact that the existence of the limit in (2) had been established before. Thus, our original discussion of the existence of the limit in (2) for real or complex x can be dispensed with entirely.

3.4 Euler's formula: is there an intuitive background?

The usual proof of Euler's formula (15) relies on the Taylor expansions of $\sin x$, $\cos x$, and $\exp(x)$ (also written as $\exp x$; its Taylor expansion is the series in (5)). This proves that Euler's formula is a consequence of (2) (or, more directly, the series expansion of $\exp(x)$ given in (5)), but it leaves one completely mystified as to what Euler's formula has to do with what one normally considers exponentiation. The proof of Euler's formula given above appears to provide a natural intuitive connection of Euler's formula with the multiplication of complex numbers.