First order partial differential equations*

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1 The general first order partial differential equation

The general first order partial differential equation with independent n variables x_1, x_2, … x_n and unknown function u can be written in the form

$$F(x_1, x_2, \dots, x_n, u, p_1, p_2, \dots, p_n), \tag{1}$$

where $p_i = \partial u/\partial x_i$. We will assume that F is continuously differentiable in all its variables,[1] and that $\sum_{i=1}^{n} F_{p_i}^2 \neq 0$; since we are dealing with real numbers only, the latter is simply an abbreviated way of saying that not all of the partial derivatives F_{p_i} are zero, i.e., (1) is an equation that involves at least one of the partial derivatives p_i at every point.

The solution of equation (1) is a function $u = \phi(x_1, x_2, \dots, x_n)$ such that the equation

$$F(x_1, x_2, \dots, \phi, \phi_{x_1}(x_1, \dots, x_n), \phi_{x_2}(x_1, \dots, x_n), \dots, \phi_{x_n}(x_1, \dots, x_n)) = 0$$

is satisfied in an *region*[2] of the n-dimensional space $\mathbb{R}^n$.[3] We will look for the solutions of equation (1) with the aid of characteristic curves, which will be curves χ: $x_i = x_i(s)$, $p_i = p_i(s)$, $1 \le i \le n$, and $u = u(s)$, $s \in I$ in the space $\mathbb{R}^{2n+1}$, where I is an interval in $\mathbb{R}$ such that if one point of this curve

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[1] We will try to keep this discussion on the intuitive level, without including strictly rigorous proofs of all the statements, so we may not always be all that careful about the exact assumptions needed for the validity of all the results we state.

[2] A region is a nonempty connected open set.

[3] $\mathbb{R}$ denotes the set of real numbers, and $\mathbb{R}^n$ denotes the set of all n-tuples $(x_1, x_2, \dots, x_n)$ of real numbers.

is contained in a solution of equation (1) then the whole curve χ lies in this solution. Assuming we have such a curve, differentiate the equation

$$F(x_1(s), x_2(s), \dots, x_n(s), u(s), p_1(s), p_2(s), \dots, p_n(s)) = 0$$

with respect to s. We obtain

$$0 = \frac{dF}{ds} = \sum_{i=1}^{n} F_{x_i}\frac{dx_i}{ds} + F_u\frac{du}{ds} + \sum_{i=1}^{n} F_{p_i}\frac{dp_i}{ds} = \sum_{i=1}^{n}\left((F_{x_i} + F_u p_i)\frac{dx_i}{ds} + F_{p_i}\frac{dp_i}{ds}\right), \tag{2}$$

where, to obtain the second equation, we used that

$$\frac{du}{ds} = \sum_{i=1}^{n} u_{x_i}\frac{dx_i}{ds} = \sum_{i=1}^{n} p_i\frac{dx_i}{ds} \tag{3}$$

We can satisfy equation (2) by taking

$$\frac{dx_i}{ds} = F_{p_i}, \qquad \frac{dp_i}{ds} = -F_{x_i} - F_u p_i. \tag{4}$$

Substituting the first equation here into (3), we obtain

$$\frac{du}{ds} = \sum_{i=1}^{n} p_i F_{p_i}. \tag{5}$$

Equations (4) and (5) are the ordinary differential equations for a characteristic curve χ.

For simplicity, first consider the case $n = 2$. In this case, characteristic curve is a curve in $\mathbb{R}^5$; a point on this curve is (x_1, x_2, p_1, p_2, u). The point (x_1, x_2, u) in $\mathbb{R}^3$ here represent a point of the solution surface in which the characteristic curve lies. The vector $\langle p_1, p_2, -1\rangle$ is perpendicular to the solution surface. Thus, the numbers p_1 and p_2 describe the tangent plane of the solution surface at the point (x_1, x_2, u). Thus, the characteristic curve can be thought of as a curve in $\mathbb{R}^3$ with a tangent plane fitted at each point of this curve. For this reason, the characteristic curve is often called a *characteristic strip*. For an arbitrary $n \geq 2$, instead of thinking about a characteristic curve as a curve in $\mathbb{R}^{2n+1}$, one can think about it as a *characteristic strip* in $\mathbb{R}^{n+1}$.

1.1 The initial-value problem

The initial-value problem, or Cauchy problem, is to find a solution $u = \phi(x_1, x_2, \dots, x_n)$ of equation (1) containing an $n-1$-dimensional initial *strip manifold*.[4] To describe the initial value problem, we require with given functions f_i, g_i, and f that for the $2n+1$ tuple $(x_1, \dots, x_n, u, p_1, \dots, p_n)$ with $x_i = f_i(t_1, \dots, t_{n-1})$, $p_i = g_i(t_1, \dots, t_{n-1})$, $u = f(t_1, \dots, t_{n-1})$, where $(t_1, \dots, t_{n-1})$ belong to a region Σ of $\mathbb{R}^{n-1}$, the equation $u = \phi(x_1, x_2, \dots, x_n)$ and $(\partial\phi/\partial x_i)(x_1, x_2, \dots, x_n) = p_i$ be satisfied. These initial values allow us to solve the characteristic equations (4) and (5) for functions $x_i = x_i(s, t_1, \dots, t_n)$, $p_i = p_i(s, t_1, \dots, t_n)$, and $u_i = u_i(s, t_1, \dots, t_n)$ such that the

[4] Manifold is a technical term in mathematics; we will avoid a precise definition of the term. To give an idea, in the three-dimensional space $\mathbb{R}^3$ a two-dimensional manifold is a "nice" surface ("nice" means smooth in some sense), and in a one-dimensional manifold is a nice curve. A one-dimensional strip-manifold could be thought of as a thin strip of a surface; actually, no surface is given, but at any point of the curve a tangent plane is given. Then this one-dimensional strip manifold will lie in a surface if the curve itself lies in the surface, and at each point of the curve of the given tangent plane is tangent to the surface.

initial conditions $x_i(0, t_1, \ldots, t_n) = f_i(t_1, \ldots, t_n - 1)$, $p_i(0, t_1, \ldots, t_n) = g_i(t_1, \ldots, t_n - 1)$, and $u(0, t_1, \ldots, t_n) = f(t_1, \ldots, t_n - 1)$ are satisfied, assuming that the functions occurring in these equations are nice enough to ensure that with open interval I containing the point $s = 0$, the solution of these equations exists and is unique in the set I.

The functions f_i, g_i, and f need to satisfy certain *consistency* conditions.[5] To ensure that $F = 0$ for $s = 0$ we need that

$$\begin{aligned} F(f_1(t_1, \ldots, t_{n-1}) \ldots, f_n(t_1, \ldots, t_{n-1}),& \\ f(t_1, \ldots, t_{n-1}), g_1(t_1, \ldots, t_{n-1}), \ldots, g_n(t_1, \ldots, t_{n-1})) = 0,& \end{aligned} \tag{6}$$

and to ensure that the equation $\partial u/\partial x_i = p_i$ is not contradicted by the initial conditions at $s = 0$ we need that

$$\frac{\partial f}{\partial t_i} = \sum_{j=1}^{n-1} g_j \frac{\partial f_j}{\partial t_i} \quad \text{for} \quad (i = 1, \ldots, n-1). \tag{7}$$

These equations say that

$$\frac{\partial u}{\partial t_i} = \sum_{j=1}^{n-1} p_j \frac{\partial x_j}{\partial t_i} \quad \text{for} \quad (i = 1, \ldots, n-1).$$

holds at $s = 0$, which is just the chain rule, assuming that we indeed have indeed $\partial u/\partial x_j = p_j$ at $s = 0$.

Another condition that needs to be satisfied is that for $s = 0$ and for any $(t_1, t_2, \ldots, t_{n-1})$ is the initial manifold Σ the determinant

$$\begin{vmatrix} \frac{\partial x_1}{\partial s} & \frac{\partial x_2}{\partial s} & \cdots & \frac{\partial x_n}{\partial s} \\ \frac{\partial x_1}{\partial t_1} & \frac{\partial x_2}{\partial t_1} & \cdots & \frac{\partial x_n}{\partial t_1} \\ \frac{\partial x_1}{\partial t_2} & \frac{\partial x_2}{\partial t_2} & \cdots & \frac{\partial x_n}{\partial t_2} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial x_1}{\partial t_{n-1}} & \frac{\partial x_2}{\partial t_{n-1}} & \cdots & \frac{\partial x_n}{\partial t_{n-1}} \end{vmatrix}$$

be nonzero. This means that the row-vectors of this matrix are linearly independent, i.e., that they span an n-dimensional vector space.[6] In other words, this means that initial manifold is indeed $n-1$ dimensional, and that tangent vector of the characteristic curve (represented by the first row of this matrix) is outside this manifold. Using the first group of the characteristic equations (4), this means

[5] *Consistent* in the technical sense used in mathematics means "free from contradiction." Thus, these conditions ensure that the specifications of the initial values do not contradict the differential equation (1).

[6] If $\langle a_1, a_2, a_3 \rangle$, $\langle b_1, b_2, b_3 \rangle$, and $\langle c_1, c_2, c_3 \rangle$ are position vectors in $\mathbb{R}^3$, then the determinant

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

represents the volume of the parallelepiped determined by these vectors, showing that this determinant is zero exactly if these vectors lie in a plane.

that we must have

$$
\tag{8}
\begin{vmatrix}
F_{p_1} & F_{p_2} & \dots & F_{p_n} \\
\frac{\partial x_1}{\partial t_1} & \frac{\partial x_2}{\partial t_1} & \dots & \frac{\partial x_n}{\partial t_1} \\
\frac{\partial x_1}{\partial t_2} & \frac{\partial x_2}{\partial t_2} & \dots & \frac{\partial x_n}{\partial t_2} \\
\dots & \dots & \dots & \dots \\
\frac{\partial x_1}{\partial t_{n-1}} & \frac{\partial x_2}{\partial t_{n-1}} & \dots & \frac{\partial x_n}{\partial t_{n-1}}
\end{vmatrix} \neq 0 \quad \text{for} \quad s = 0 \quad \text{at each} \quad (t_1, \dots, t_{n-1}) \in \Sigma.
$$

While our discussion explains the method used in the following example, there are gaps in our discussion. For example, it nees to be proved that using the characteristic that the p_i obtained by the characteristic equations (4) and (4) used to solve an initial-value problem indeed satisfy the requirement that $p_i = \partial u/\partial x_i$ (see e.g. [2, p. 29] or [1, p. 81]).

Example 1. Solve the equation $u_z = u_x^2 + u_y^2$ with the initial condition $u = x^2 + y^2$ at the point (x, y, z) with $z = 0$.

Solution. We have $F(x, y, z, u, p, q, r) = p^2 + q^2 - r$, where the differential equation is $F = 0$ and $p = \partial u/\partial x$, $q = \partial u/\partial y$, and $r = \partial u/\partial z$. The initial manifold is given by

$$
\tag{9}
x = t_1, \quad y = t_2, \quad \text{and} \quad z = 0.
$$

and the initial condition on this manifold is

$$
\tag{10}
u = t_1^2 + t_2^2.
$$

The variables p, q, r are not specified by the initial conditions, since the consistency conditions are sufficient to determine their initial values; namely, equation (7) gives

$$
\tag{11}
2t_1 = p, \qquad \text{and} \qquad 2t_2 = q,
$$

and then $r = 4t_1^2 + 4t_2^2$ according to (6), even though these equations will not be used.

The characteristic equations given in (4) become

$$
\frac{dx}{ds} = 2p \; (= F_p), \quad \frac{dy}{ds} = 2q \; (= F_q), \quad \frac{dz}{ds} = -1 \; (= F_r), \quad \frac{dp}{ds} = \frac{dq}{ds} = \frac{dr}{ds} = 0.
$$

The fact that u does not directly occur in F makes these equations much simpler. Furthermore, characteristic equation (5) becomes

$$
\frac{du}{ds} = p \cdot 2p + q \cdot 2q + r \cdot (-1) = 2p^2 + 2q^2 - r = p^2 + q^2,
$$

where the last equation holds since $p^2 + q^2 - r = 0$ according to the equation $F = 0$. Since p, q, and r do not depend on s according to the first group of equations, these equations are easily solvable. Also using equations (11), we obtain

$$
x = 2ps + c_1 = 4t_1 s + c_1, \quad y = 2qs + c_2 = 4t_2 s + c_2, \quad \text{and} \quad z = -s + c_3
$$

and by the characteristic equation for u we obtain

$$
u = (p^2 + q^2)s + c_3 = (4t_1^2 + 4t_2^2)s + c_4.
$$

For $s = 0$ these equations must agree with equations (9) and (10). That is, we have $c_1 = t_1$, $c_2 = t_2$, $c_2 = 0$ and $c_4 = t_1^2 + t_2^2$, and so

$$x = (4s+1)t_1, \quad y = (4s+1)t_2, \quad z = -s, \quad \text{and} \quad u = (4s+1)(t_1^2 + t_2^2).$$

t_1, t_2, and s can easily eliminated from these equations, and we obtain the solution in explicit form:

$$u = \frac{x^2 + y^2}{1 - 4z}.$$

2 Separation of variables and the complete integral

2.1 Separation of variables

One can occasionally solve a partial differential equation by making an *ansatz*[7] seeking the solution of the equation in the form $u(x, y) = u(x) + u(y)$ or $u(x, y) = u(x)u(y)$. Solving the equation by making such an ansatz is called *separation of variables*.

Example 2. Find the solution of the equation $u_x^2 + u_2^2 = 1$.

Solution. Assume $u(x, y) = u_1(x) + u_2(x)$. Then $u_x(x, y) = u_1'(x)$ and $u_y(x, y) = u_2'(y)$; hence the equation can be written as $(u_1'(x))^2 = 1 - (u_2'(y))^2$. Since the left-hand side depends only on x and the right-hand side, only on y, both sides must be constant. So there is a constant a such that $(u_1'(x))^2 = a^2$ ($|a| \le 1$), so we may assume that $u'(x) = a$.[8] We then have $u_1(x) = ax + b_1$ where a and b_1 are arbitrary constants except that $|a| \le 1$. We also have $(u_2'(y))^2 = 1 - a^2$, and so $u_2'(y) = \pm\sqrt{1 - a^2}$. Hence $u_2'(y) = \pm\sqrt{1 - a^2}y + b_2$. As $u(x, y) = u_1(x) + u_2(x)$, writing $b = b_1 + b_2$, we obtain the solution of the equation

$$u(x, y) = ax \pm \sqrt{1 - a^2}y + b,$$

where a and b are arbitrary constants.

A solution of a first order partial differential equation with n independent variables is containing n arbitrary constants is called a *complete integral* of the equation,[9] discussed below in more detail in case $n = 2$. Thus, the solution we obtained in the above example is a complete integral. In fact, the method of separation of variables usually gives the solution of a first order partial differential equation in the form of a complete integral.

Example 3. Solve the equation $u = u_x u_y$.

[7] *Ansatz* (plural *Ansätze*) is a German noun that is often encountered in the mathematics and physics literature, In German, all nouns are capitalized, but in English one does not need to capitalize the word. In the mathematical literature it usually means a hypothesis, a setup, or a starting point, most often an assumption about the form in which the solution of a differential equation is to be sought. See the Wikipedia article

`http://en.wikipedia.org/wiki/Ansatz`

[8] Actually, only $u_1'(x) = \pm a$ follows, but we may chose the sign of a appropriately so as to satisfy the equation $u_1'(x) = 1$.

[9] The constants involved must affect the solution in an *independent* way, a concept we will explain precisely in case $n = 2$ below.

Solution. Assume $u(x, y) = u_1(x)u_2(y)$. Then $u_x(x, y) = u_1'(x)u_2(x)$ and $u_y(x, y) = u_1(x)u_2'(x)$. Hence $u_1(x)u_2(y) = u_1'(x)u_2'(x) \cdot u_1(x)u_2(x)$. Ignoring the (relatively uninteresting) case when $u_1'(x)u_2'(x) = 0$, we obtain $u_1'(x) = 1/u_2'(y)$. The left-hand side depends only on x, and the right-hand side, only on y, so both sides must be constant. This means that $u_1'(x) = C$ and $u_2'(x) = 1/C$ for an arbitrary (nonzero) constant C. Hence $u_1(x) = Cx+A$ and $u_2(y) = (1/C)y+B$ with arbitrary constants A and B. So $u(x, y) = u_1(x)u_2(y) = (Cx+A)((1/C)y+B) = xy+(BC)x+(A/C)y+AB$. Writing $a = BC$ and $b = A/C$, we have $AB = ab$, so we obtain the solution

$$u(x, y) = xy + ax + by + ab.$$

This is again a complete integral of the equation.

Example 4. Solve the equation $u = u_x x + u_y y$.

Solution. The equation in the problem is a special case of *Clairaut*'s equation. We are looking for the solution in the form $u(x, y) = u_x(x) + u_y(y)$. The equation becomes

$$u_1(x) - xu_1'(x) = -(u_2(y) - yu_2'(y)). \tag{12}$$

The left-hand side depends only on x, and the right-hand side, only on y, so the two sides must be constant; denote this constant by $-C$. After dividing through by x, the left-hand side being equal to C gives the equation

$$u_1(x) - \frac{1}{x}u_1(x) = \frac{C}{x}.$$

This is a linear first order ordinary differential equation, the general form of which is

$$y' + P(x)y = Q(x).$$

The solution of this equation can be written as

$$y = e^{-\int P(x)\,dx}\left(\int e^{\int P(x)\,dx} Q(x)\,dx + c\right),$$

where c is an arbitrary constant. Thus, we obtain

$$u_1(x) = e^{\log x}\left(\int e^{-\log x}\frac{C}{x}\,dx + a\right) = x\left(\int Cx^{-2}\,dx + a\right) = x(-Cx^{-1} + a) = -C + ax,$$

where a is an arbitrary constant. Similarly, the right-hand side of equation (12) being equal to C, we obtain that

$$u_2(y) - \frac{1}{y}u_2(y) = -\frac{C}{y}.$$

The solution of this equation is

$$u_2(y) = e^{\log y}\left(-\int e^{-\log y}\frac{C}{y}\,dy + b\right) = y\left(-\int Cy^{-2}\,dy + b\right) = y(Cy^{-1} + b) = C + by,$$

where b is an arbitrary constant. Hence we obtain $u(x, y) = ax + by$ with a and b arbitrary constant. This is again a complete integral of the equation above.

2.2 The envelope of a family of curves

To simplify the discussion of the complete integral below, we first describe what is meant by the envelope of a family of curves $y = f(x, a)$; this equation describes a curve for each value of the parameter a. The envelope to this family will be a curve that is tangent to each member of the family at some point. To determine such an envelope, find a as a function $a = a(x)$ from the equation $f_a(x, a) = 0$, assuming such an a can be uniquely determined. Put $\psi(x) = f(x, a(x))$. Then the curve $y = \psi(x)$ is tangent to the curve $f(x, a(x_0))$ at the point $x = x_0$. Indeed,

$$\psi'(x) = f_x(x, a(x)) + f_a(x, a(x))a'(x) = f_x(x, a(x)), \tag{13}$$

since we have $f_a(x, a(x)) = 0$ according to the definition of $a(x)$. That $\psi'(x) = f_x(x, a(x_0))$ in case $x = x_0$. If $y = f(x, a)$ represents a family of solutions of a differential equation $F(x, y, y') = 0$, then the envelope $y = \psi(x)$ is also a solution of this equation, according to (13).

2.3 The complete integral

We confine our discussion to the case $n = 2$, that is, instead of (1) we will consider the differential equation

$$F(x, y, u, p, q) = 0, \tag{14}$$

where $p = \partial u/\partial x$ and $q = \partial u/\partial y$, and we assume that $F_p^2 + F_q^2 \neq 0$. A complete integral of this equation of the form

$$u = \phi(x, y, a, b),$$

where a and b are arbitrary parameters affecting the solutions *independently*, i.e., in a way that it is impossible to replace a and b by a single parameter. The independence of the parameters can be precisely expressed by saying that the rank of the matrix

$$\begin{bmatrix} \phi_a & \phi_{ax} & \phi_{ay} \\ \phi_b & \phi_{bx} & \phi_{by} \end{bmatrix}$$

is 2.[10] The complete integral plays a very important role in the Hamilton–Jacobi partial differential equation of classical mechanics.

[10] This condition expresses the fact that it is not possible to find a function $g(a, b)$ such that we can write $\phi(x, y, a, b) = \psi(x, y, g(a, b))$, i.e., that it is not possible to replace the two parameters with the single parameter $c = g(a, b)$. Indeed, if there were such a function $g(a, b)$ then it is not hard to show by using the chain rule is solvable for x and y for any a and b sufficiently close to a_0 and b_0. is solvable for x and y for any a and b sufficiently close to a_0 and b_0. that the two rows of the above matrix would be linearly dependent, and hence its rank would be 1.

The considerations get somewhat simplified if instead one makes the assumption

$$\begin{vmatrix} \phi_{ax} & \phi_{ay} \\ \phi_{bx} & \phi_{by} \end{vmatrix} \neq 1.$$

According to a result known as the *Implicit Function Theorem* or *Implicit Mapping Theorem*, this implies the following. If for some choices of x_0, y_0, a_0, b_0, A, and B we have

$$\phi_a(x_0, y_0, a_0, b_0) = A,$$
$$\phi_b(x_0, y_0, a_0, b_0) = A,$$

then the system of equations

$$\phi_a(x, y, a, b) = A,$$
$$\phi_b(x, y, a, b) = B,$$

is solvable for x and y for any a and b sufficiently close to a_0 and b_0.

The complete integral is not the general solution of a partial differential equation, since we have seen that the general solution of a first order partial differential equation with two independent variables involves an arbitrary function. It is possible, however, to use the complete integral to obtain the general solution as an envelope of a family of solutions given by the complete integral. To see this, let $w(a)$ be an arbitrary differentiable function, and consider the envelope of the family of solutions $\phi(x, y, a, w(a))$ of (14), where a is a parameter. Analogously to the discussion of envelope above, we can find the envelope of this family by first noting that

$$\frac{\partial\phi(\phi(x, y, a, w(a))}{\partial a} = \phi_a(x, y, a, w(a)) + w'a\phi_b(x, y, a, w(a)) = (\phi_a + w'(a)\phi_b)(x, y, a, w(a)),$$

where the right-hand side is just an abbreviated way of writing the middle member of these equations, and then solving the equation

$$(\phi_a + w'(a)\phi_b)\,(x, y, a, w(a)) = 0 \tag{15}$$

for a as a function of $a = a(x, y)$ of x and y, and then writing

$$\psi(x, y) = \phi(x, y, a(x, y), w(a(x, y))). \tag{16}$$

It is easy to verify that $\psi(x, y)$ is a solution of equation (14). Indeed, we with $u = \psi(x, y)$ we have

$$\begin{aligned}
u(x_0, y_0) &= \phi(x_0, y_0, a(x_0, y_0), w(a(x_0, y_0))),\\
u_x(x_0, y_0) &= \big(\phi_x + \phi_x a_x(x_0, y_0) + \phi_x\phi_b w'(a(x_0, y_0))a_x(x_0, y)\big)\,(x_0, y_0, a(x_0, y_0), w(a(x_0, y_0))),\\
&= \phi_x\,(x_0, y_0, a(x_0, y_0), w(a(x_0, y_0))),\\
u_y(x_0, y_0) &= \big(\phi_y + \phi_y a_y(x_0, y_0) + \phi_y\phi_b w'(a(x_0, y_0))a_x(x_0, y)\big)\,(x_0, y_0, a(x_0, y_0), w(a(x_0, y_0))),\\
&= \phi_y\,(x_0, y_0, a(x_0, y_0), w(a(x_0, y_0))).
\end{aligned}$$

Since $u = \phi(x, y, a, b)$ satisfies (14) for any a, b at every point (x, y), it in particular satisfies this equation for $a = a(x_0, y_0)$ and $b = b(x_0, y_0)$ at the point (x_0, y_0). Hence the above equations show that $u = \psi(x_0, y_0)$ also satisfies (14) at the point (x_0, y_0). (x_0, y_0) being arbitrary, this shows that $u = \psi(x, y)$ indeed satisfies equation (14).

The solution $\phi(x, y)$ depends on the arbitrary function w, and so it represents the general solution of (14). We mentioned earlier that the general solution does not necessarily represent all solutions of the equations. In particular, another solution, called the singular solution can be obtained from the complete integral by taking the envelope of the whole family $\phi(x, y, a, b)$. To do this, solve the system of equations

$$\begin{aligned}
\phi_a(x, y, a, b) &= 0,\\
\phi_b(x, y, a, b) &= 0
\end{aligned}$$

for a and b in terms of x and y to obtain $a = a(x, y)$ and $b = a(x, y)$, and take $\sigma(x, y) = \phi(x, y, a(x, y), b(x, y))$. It is easy to show that $u = \sigma(x, y)$ is then a solution of (14). This solution is called the *singular solution*, usually not obtained as one of the solutions represented by the general solution.

2.4 Determining the characteristic strips from the complete integral

If two solutions Φ and Ψ of equation (14) agree and tangent at the point (x_0, y_0), that is, if $\Phi(x_0, y_0) = \Psi(x_0, y_0)$, $\Phi_x(x_0, y_0) = \Psi_x(x_0, y_0)$, and $\Phi_y(x_0, y_0) = \Psi_y(x_0, y_0)$, then, assuming that the solution of the characteristic equations (the equations analogous to equations (4) and (5) in case $n = 2$) are unique, the characteristic strip (x, y, p, q) ($p = u_x$ and $q = u_y$) going through the point $(x_0, y_0, u_0, p_0, q_0)$, where $u_0 = \Phi(x_0, y_0)$, $p_0 = \Phi_x(x_0, y_0)$, and $q_0 = \Phi_y(x_0, y_0)$, must lie in both solutions Φ and Ψ (at least locally, i.e., in a small region where the uniqueness of the solution of the characteristic equations is satisfied). Unless the two solutions agree in a small region, they can only agree on the characteristic strip, and nowhere else (again, locally, i.e., in a small region – anything may happen outside a small region containing the point (x_0, y_0, u_0)); this again follows from the uniqueness of the characteristic strip. Therefore, we may try to find the characteristic strip going through the point $(x_0, y_0, u_0, p_0, q_0)$ by taking the intersection of the solutions Φ and Ψ.

Now, for a given a, the solutions $\phi(x, y, a, w(a))$ and $\psi(x, y)$ given by (16) agree and are tangent at a point (x_0, y_0) for which $a = a(x_0, y_0)$; the set of these points (x_0, y_0) will be part of a characteristic strip, along with the corresponding other coordinates u_0, p_0, and q_0. To find these points, we need to find such (x_0, y_0) pairs by solving equation (15) for (x, y) with the given a. For this, we do not even need to know the whole of the function w, we only need to know the values $w(a)$ and $w'(a)$ for the given a. Writing $b = w(a)$ and $-\eta = w'(a)$, this means that the characteristic strip can be found by solving the equation

$$(\phi_a - \eta\phi_b)\,(x, y, a, b) = 0 \tag{17}$$

for x and y, given a, b, and η, and then finding u, p, and q from the equations $u = \phi(x, y, a, b)$, $p = \phi_x(x, y, a, b)$, and $p = \phi_y(x, y, a, b)$. Using this way of finding characteristic strips, we can solve initial value problems by fitting a characteristic strip to each point of the initial curve.

Example 5. Find the solution of the equation $u = u_x x + u_y y$ with initial conditions C: $x + y = 1$, $u = x^2$.

Solution. Write the initial curve C in parametric form as $x = t$, $y = 1-t$, and $u = t^2$. The complete integral of the given partial differential equation is $u = \phi(x, y, a, b) = ax + by$ according to Example 4, so $p = \phi_x = a$ and $q = \phi_y = b$. According to the consistency condition $du/dt = 2t$ we have $2t = a + (-b)$ (cf. (7)), i.e., $b = a - 2t$. Hence initial conditions give

$$t^2 = \phi(t, 1-t, a, b) = \phi(t, 1-t, a, a-2t) = at + (a-2t)(1-t) = a - 2t + 2t^2$$

(cf. (6), i.e., $a = 2t - t^2$, and so $b = a - 2t = -t^2$.

The equation of a characteristic is $x - \eta y = 0$ (cf. (17)), i.e., $\eta = x/y$. Requiring that this characteristic passes through the point $(x, y, u) = (t, 1-t, t^2)$ of the curve C, this gives $\eta = t/(t-1)$. That is, the equation of the characteristic is $x - \eta y = 0$ with this value of η; that is, the characteristic is $x(1-t) = y$, i.e., $x = t(x + y)$.

To sum up, $t = x/(x + y)$, $u = ax + by$, $a = 2t - t^2$, and $b = -t^2$. Therefore, the solution is

$$u(x, y) = \left(\frac{2x}{x+y} - \frac{x^2}{(x+y)^2}\right)x - \frac{x^2}{(x+y)^2}y = \frac{x^2}{x+y}.$$

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