The Frenet–Serret formulas

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1 The Frenet–Serret frame of a space curve

We will consider smooth curves given by a parametric equation in a three-dimensional space. That is, writing bold-face letters of vectors in three dimension, a curve is described as \( \mathbf{r} = \mathbf{F}(t) \), where \( \mathbf{F}' \) is continuous in some interval \( I \); here the prime indicates derivative. The length of such a curve between parameter values \( t_0 \in I \) and \( t_1 \in I \) can be described as

\[
\sigma(t_1) = \int_{t_0}^{t_1} |\mathbf{F}'(t)| \, dt = \int_{t_0}^{t_1} \left| \frac{d\mathbf{r}}{dt} \right| \, dt
\]

where, for a vector \( \mathbf{u} \) we denote by \( |\mathbf{u}| \) its length; here we assume \( t_0 \) is fixed and \( t_1 \) is variable, so we only indicated the dependence of the arc length on \( t_1 \). Clearly, \( \sigma \) is an increasing continuous function, so it has an inverse \( \sigma^{-1} \); it is customary to write \( s = \sigma(t) \). The equation

\[
\mathbf{r} = \mathbf{F}(\sigma^{-1}(s)) \quad s \in J \overset{\text{def}}{=} \{ \sigma(t) : t \in I \}
\]

is called the re-parametrization of the curve \( \mathbf{r} = \mathbf{F}(t) \ (t \in I) \) with respect to arc length. It is clear that \( J \) is an interval. To simplify the description, we will always assume that \( \mathbf{r} = \mathbf{F}(t) \) and \( s = \sigma(t) \),

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so we will just use the variables \( r, t, \) and \( s \) instead of using function notation. We will use prime to indicate the differentiation \( d/dt \), while the differentiation \( d/ds \) will not be abbreviated. We will assume that \( r' \neq 0 \) for any \( t \in I \); then \( r' \) is a tangent vector to the curve corresponding to the given parameter value \( t \).

By the fundamental theorem of calculus, equation (1) implies
\[
 s' = \frac{ds}{dt} = \frac{d\sigma(t)}{dt} = |F'(t)| = |r'|. 
\]
Hence, by the chain rule of differentiation we have
\[
 \frac{d}{ds} = \frac{dt}{ds} \frac{d}{dt} = \left( \frac{1}{\frac{ds}{dt}} \right) \frac{d}{dt} = \frac{1}{|r'|} \frac{d}{dt}, 
\]
where the second equation follows from equation (1) by the fundamental theorem of calculus. When using differential operator notation as in \( d/ds \), everything after the differential operator up to the next + or − sign needs to be differentiated; the expression preceding the differential operator is not to be differentiated. The unit tangent vector \( T \) is defined as
\[
 T \overset{\text{def}}{=} \frac{1}{|r'|} r' = \frac{dr}{ds};
\]
the second equation follows from equations (4). The curvature is defined as
\[
 \kappa \overset{\text{def}}{=} \left| \frac{dT}{ds} \right|. 
\]
We will assume that \( \kappa \neq 0 \). The unit normal vector is defined as
\[
 N \overset{\text{def}}{=} \frac{1}{\kappa} \frac{dT}{ds}. 
\]
Note that \( |T|^2 = T \cdot T \), so by the product rule of differentiation,
\[
 \frac{dT}{ds} \cdot T = \frac{1}{2} \left( \frac{dT}{ds} \cdot T + T \cdot \frac{dT}{ds} \right) = 0; 
\]
hence \( T \) is perpendicular to \( dT/ds \), and so \( N \) is perpendicular to \( T \). The unit binormal vector is defined as
\[
 B \overset{\text{def}}{=} T \times N. 
\]
The vectors \( T, N, B \) form the basic unit vectors of a coordinate system especially useful for describing the local properties of the curve at the given point. These three vectors form what is called the Frenet–Serret frame. Equation (9) implies that the vectors \( T, N, B \) form a right-handed system of pairwise perpendicular unit vectors. Any cyclic permutation of these vectors also form a right-handed system of pairwise perpendicular unit vectors; therefore we have
\[
 T \times N = B, 
\]
\[
 N \times B = T, 
\]
\[
 B \times T = N. 
\]

1. If the equation \( r = F(t) \) describes a moving point, where \( t \) is time, then \( r \) is the velocity vector of the moving point at time \( t \). That is, the length of \( r' \) is its speed, while the direction of \( r' \) is its direction of its movement. If \( r' = 0 \), then the point stopped moving at the given time, and then it may resume its movement in a different direction. This means that even though the function \( F \) is differentiable, the tangent line to the curve described by the function may not be defined at this point.

2. A unit vector is a vector of length 1.

3. We will comment later on what happens when \( \kappa \neq 0 \).
2 The Frenet–Serret formulas

As $|N| = 1$, we have $|N|^2 = N \cdot N = 1$, and so, similarly to equation (8), we have

$$\frac{dN}{ds} \cdot N = 0.$$ 

That is $dN/ds$ is perpendicular to $N$, so we have

$$\frac{dN}{ds} = \alpha T + \tau B$$

for some numbers $\alpha$ and $\tau$ (depending on $t$). Here $\tau$ is called the torsion of the curve at the point; the value of $\alpha$ will be determined below. Using this equation, equations (10), and the product rule of differentiation for vector products, we have

$$\frac{dB}{ds} = \frac{d}{ds}(T \times N) = \frac{dT}{ds} \times N + T \times \frac{dN}{ds} = \kappa N \times N + T \times (\alpha T + \tau B) = \tau T \times B = -\tau N;$$

the third equation follows by equations (6) and (11). The first term of the third member is zero; so is first term after distributing the cross product in the second term in the third member. Thus the fourth equation follows; to obtain the last equation we used equation (10). Hence, using equation (10), we have

$$\frac{dN}{ds} = \frac{d}{ds}(B \times T) = \frac{dB}{ds} \times T + B \times \frac{dT}{ds} = -\tau N \times T + B \times (\kappa N) = \tau B + \kappa B \times N = -\kappa T + \tau B;$$

This equation shows that $\alpha$ in equation (11) equals $-\kappa$; however, this fact and equation (11) itself is no longer of any interest, since this equation is subsumed in the last equation. Equations (7), (12), and (13) are called the Frenet–Serret formulas. To summarize these formulas, we have

$$\frac{dT}{ds} = \kappa N,$$

$$\frac{dN}{ds} = -\kappa T + \tau B,$$

$$\frac{dB}{ds} = -\tau N.$$

3 The first three derivatives of $r$

As we mentioned above, we will indicate derivation with respect to $t$ by prime. According to equations (5) and (3), we have

$$r' = |r'| T = s'T.$$
We can do further differentiations with respect to \( t \) by using equations (14), (4), and (3). We have
\[
(16) \quad r'' = s'' T + s'T' = s'' T + (s')^2 \frac{dT}{ds} = s'' T + (s')^2 \kappa N.
\]

Further, repeatedly using equations (15) and (14), we have
\[
(17) \quad r''' = s''' T + s'' T' + (2s's'' \kappa + (s')^2 \kappa') N + (s')^2 \kappa N'
\]
\[
= s''' T + s'' s \frac{dT}{ds} + (2s's'' \kappa + (s')^2 \kappa') N + (s')^3 \kappa \frac{dN}{ds}
\]
\[
= s''' T + s'' s \kappa N + (2s's'' \kappa + (s')^2 \kappa') N + (s')^3 \kappa (-\kappa T + \tau B)
\]
\[
= (s''' - (s')^3 \kappa) T + (3s'' s \kappa + (s')^2 \kappa') N + (s')^3 \kappa \tau B.
\]

It is now easy to express \( \kappa \) and \( \tau \) in terms of derivatives with respect to \( t \). Equations (15), (16), and (10) give
\[
(18) \quad r' \times r'' = (s')^3 \kappa B,
\]
so using equation (3), we obtain that
\[
(19) \quad \kappa = \frac{|r' \times r''|}{|r'||^3}
\]
by noting that \( |B| = 1 \). By equations (18) and (17) we have
\[
(r' \times r'') \cdot r''' = (s')^6 \kappa^2 \tau.
\]
Hence, using equation (18) and noting that \( |B| = 1 \), we obtain that
\[
(20) \quad \tau = \frac{(r' \times r'') \cdot r'''}{|r' \times r''|^2}.
\]

4 Examples and discussion

Since the curvature \( \kappa \) and the torsion \( \tau \) are defined in terms of the local coordinate frame \( T, N, B \) in arc-length parametrization, they only depend on the shape of the curve and not on the choice of the coordinate system \( x, y, z \) and the choice of the parameter \( t \). Hence, for a curve that we want to calculate the curvature or the torsion, we may set up the coordinate system \( x, y, z \) and choose a parametrization that make these calculations especially simple.

4.1 The curvature of a circle

We consider the circle of radius \( R > 0 \) lying in the \( x, y \) plane, and centered at the origin. This circle can be parametrized by the equation
\[
r = R(i \cos t + j \sin t),
\]
where \( i, j, \) and \( k \) are the unit coordinate vectors in the directions of the positive \( x, y, \) and \( z \) axes, respectively. We have
\[
r' = R(-i \sin t + j \cos t)
\]
and 
\[ r'' = -R(i \cos t + j \sin t). \]

Hence 
\[ r' \times r'' = \begin{vmatrix} i & j & k \\ -R \sin t & R \cos t & 0 \\ -R \cos t & -R \sin t & 0 \end{vmatrix} = R^2(\sin^2 t + \cos^2 t)k = R^2k. \]

We also have 
\[ |r'| = R\sqrt{\cos^2 t + \sin^2 t} = R. \]

Hence equation (19) gives that 
\[ \kappa = \frac{R^2}{R^3} = \frac{1}{R}. \]

Thus, the curvature of a circle is the reciprocal of the radius. For this reason, given any curve, 
\[ 1/\kappa \]

is called the radius of curvature; it is the radius of the osculating circle: given an curve by the equation \( r = F(t) \), the osculating circle at a point corresponding to the parameter value \( t = t_0 \) (i.e., at the point with position vector \( r_0 = F(t_0) \)) is a circle with equation \( r = G(t) \) at the for which

\[ G(t_0) = F(t_0), \quad G'(t_0) = F'(t_0), \quad \text{and} \quad G''(t_0) = F''(t_0). \]

For the existence of such a circle, one needs to assume that \( F'(t_0) \neq 0 \) and \( F''(t_0) \neq 0 \). The case that \( F'(t_0) = 0 \) is a case of bad parametrization, when the curve may or may not have a tangent line and a curvature, but the equation is not suitable for determining the tangent line or the curvature. In this case, one needs to re-parametrize the curve in such a way that the derivative at the point with position vector \( F(t_0) \) is not zero. If \( F'(t_0) \neq 0 \) but \( F''(t_0) = 0 \), the curvature is 0, and the osculating circle degenerates into a straight line; in fact, the tangent line can be considered the osculating “circle” in this case, and one may say that the corresponding radius of curvature is infinite.

4.1 The curvature and the torsion of a helix

A helix in the standard position can be described by the equation 
\[ r = iR \cos t + jR \sin t + ctk \quad (R > 0). \]

We have 
\[ r' = -iR \sin t + jR \cos t + ck, \]
\[ r'' = -iR \cos t - jR \sin t, \]
\[ r''' = iR \sin t - jR \cos t. \]

Therefore 
\[ |r'| = \sqrt{R^2(\sin^2 t + \cos^2 t) + c^2} = \sqrt{R^2 + c^2}. \]

Furthermore, 
\[ r' \times r'' = \begin{vmatrix} i & j & k \\ -R \sin t & R \cos t & c \\ -R \cos t & -R \sin t & 0 \end{vmatrix} = iR \sin t - jR \cos t + R^2(\sin^2 t + \cos^2 t)k \]
\[ = iR \sin t - jR \cos t + R^2k. \]

4.2 Such a parametrization may not exist. If there is a suitable re-parametrization, the re-parametrization with respect to arc length will work. It is, however, possible, that the derivative of the arc-length parametrization at the given point does not exist – in which case there is no smooth re-parametrization at the point.
Thus

\[ |\mathbf{r}' \times \mathbf{r}''| = \sqrt{c^2 R^2 (\cos^2 t + \sin^2 t) + R^4} = R \sqrt{c^2 + R^2} \]

and

\[ (\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}''' = R \sqrt{c^2 + R^2} = cR^2 \sin^2 t + cR^2 \cos^2 t = cR^2. \]

Hence, using formula (19) we obtain

\[ \kappa = \frac{R \sqrt{c^2 + R^2}}{(R^2 + c^2)^{3/2}} = \frac{R}{R^2 + c^2}. \]

Similarly, using formula (20)

\[ \tau = \frac{cR^2}{R^2 (c^2 + R^2)} = \frac{c}{R^2 + c^2}. \]

An osculating curve to a given curve \( \mathbf{r} = \mathbf{F}(t) \) is a curve \( \mathbf{r} = \mathbf{G}(t) \) satisfying equations (21); these equations can also be described by saying that the curves \( \mathbf{r} = \mathbf{F}(t) \) and \( \mathbf{r} = \mathbf{G}(t) \) have a second order contact at the given point. One can generalize this to an arbitrary integer \( n \) by saying that the curve \( \mathbf{r} = \mathbf{F}(t) \) and \( \mathbf{r} = \mathbf{F}(t) \) have an order \( n \) contact for a given parameter value \( t = t_0 \) if \( \mathbf{F}'(t_0) \neq 0 \) and \( \mathbf{G}^{(k)}(t_0) = \mathbf{F}^{(k)}(t_0) \) for all \( k \) with \( 0 \leq k \leq n \).

The osculating plane is the plane spanned by the vectors \( \mathbf{T} \) and \( \mathbf{N} \). The osculating circle lies in this plane. The osculating plane has a second order contact with the curve at the point given by a parameter value \( t = t_0 \). More generally, if \( \mathbf{F}'(t_0) \neq 0 \) and \( \mathbf{F}''(t_0) \neq 0 \), then any plane curve that has a second order contact with the curve \( \mathbf{r} = \mathbf{F}(t) \) at the parameter value \( t = t_0 \) lies entirely in the osculating plane. If \( \mathbf{F}''(t_0) = 0 \) then the osculating plane is not determined since \( \kappa = 0 \) in this case so the vector \( \mathbf{N} \) is not determined (cf. equations (19) and (7) to see this). The torsion expresses the speed with which the osculating plane turns as the arc-length parameter changes (indeed, this follows from the third equation in (14), since \( \mathbf{B} \) is normal to the osculating plane).

The derivation of the Frenet–Serret formulas (14) shows the theoretical usefulness of arc-length parametrization. Re-parametrizing a curve with respect to arc-length is rarely done in practice, since the integrals involved cannot usually be evaluated, and a more useful procedure is to rewrite the formulas derived with arc-length parametrization in terms of the original parameter, as was done in formulas (19) and (20).

\footnote{This condition just mentioned may be dropped if one is only interested in the curve as a moving points. If \( F''(t_0) = 0 \) then this condition expresses only that the points near the time \( t = t_0 \) have a “higher order closeness,” but since both points stop moving at this time, this has no implication for the geometry of the two curves.}