The Laplace operator in polar coordinates in several dimensions*

Attila Máté
Brooklyn College of the City University of New York

January 15, 2015

Contents

1 Polar coordinates and the Laplacian

1.1 Polar coordinates in n dimensions

Let $n \ge 2$ be an integer, and consider the n-dimensional Euclidean space $\mathbb{R}^n$. The Laplace operator in $\mathbb{R}^n$ is $L_n = \Delta = \sum_{i=1}^n \partial^2/\partial x_i^2$. We are interested in solutions of the Laplace equation $L_n f = 0$ that are spherically symmetric, i.e., is such that f depends only on $\sum_{r=1}^n x_i^2$. In order to do this, we need to use polar coordinates in n dimensions. These can be defined as follows: for k with $1 \le k \le n$ define $r_k^2 = \sum_{i=1}^k x_i^2$ and put for $2 \le k \le n$ write

$$r_{k-1} = r_k \sin\phi_k \quad \text{and} \quad x_k = r_k \cos\phi_k. \tag{1}$$

$r_{k-1} = r_k \sin\phi_k$ and $x_k = r_k \cos\phi_k$. The polar coordinates of the point $(x_1, x_2, \dots, x_n)$ will be $(r_n, \phi_2, \phi_3, \dots, \phi_n)$.

In case $n = 2$, we can write $y = x_1$, $x = x_2$. The polar coordinates (r, θ) are defined by $r^2 = x^2 + y^2$,

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta, \tag{2}$$

so we can take $r_2 = r$ and $\phi_2 = \theta$. In case $n = 3$, the polar coordinates (r, θ, ϕ) are called spherical coordinates, and we have $y = x_1$, $x = x_2$, $z = x_3$, $r^2 = x^2 + y^2 + z^2$, $x = r\sin\phi\sin\theta$, $y = r\sin\phi\cos\theta$, and $x = r\cos\phi$, so we can take $r_3 = r$, $\phi_2 = \theta$, and $\phi_3 = \phi$. When using spherical coordinates, one often makes the restrictions $r \ge 0$, $0 \le \theta < 2\pi$, and $0 \le \phi \le \phi$, although it may occasionally be preferable to change these restrictions.

*Written for the course Mathematics 4211 at Brooklyn College of CUNY.

1.2 Cartesian and polar differential operators

The transformation $T : (x_1, x_2, \dots, x_n) \to (r_n, \phi_2, \dots, \phi_n)$ is not one-to-one, since there are several possible choices are for polar coordinates. If $P_0 = (x_1^{(0)}, x_2^{(0)}, \dots, x_n^{(0)})$ is a point for which $r_k^{(0)} \neq 0$ for any k with $2 \leq k \leq n$ then there is a neighborhood of the point P_0 in which T can be defined in a one-to-one way, and, furthermore, the transformation T so defined is differentiable any number of times. If this condition is not satisfied, then there is no neighborhood of P_0 in which T can be defined in a one-to-one way; further T is not be differentiable at the point P_0. We would like to find the partial derivatives of the inverse transformation T^{-1} of T; in order to do this, it will be easier to first determine the partial derivatives of T. Using equations (1) with $k = n$, it is easy to calculate the Jacobian matrix

$$\frac{\partial(r_{n-1}, x_n)}{\partial(r_n, \phi_n)} = \begin{bmatrix} \partial r_{n-1}/\partial r_n & \partial x_n/\partial r_n \\ \partial r_{n-1}/\partial \phi_n & \partial x_n/\partial \phi_n \end{bmatrix} = \begin{bmatrix} \sin\phi_n & \cos\phi_n \\ r_n \cos\phi_n & -r_n \sin\phi_n \end{bmatrix}.$$

Equations (1) with $k = n$ can be inverted to determine the functions $r_n = r_n(r_{n-1}, x_n)$ and $\phi_n = \phi_n(r_{n-1}, x_n)$ in a neighborhood of a point for which $r_n \neq 0$, i.e., we have $r_n \equiv r_n(r_n \sin\phi_n, r_n \cos\phi_n)$ and $\phi_n \equiv \phi_n(r_n \sin\phi_n, r_n \cos\phi_n)$ in such a neighborhood. Differentiating the latter equations with respect to r_n and ϕ_n, it follows from the chain rule of differentiation that the Jacobian matrix $\partial(r_n, \phi_n)/\partial(r_{n-1}, x_n)$ is the inverse of the matrix $\partial(r_{n-1}, x_n)/\partial(r_n, \phi_n)$. That is,

$$\frac{\partial(r_n, \phi_n)}{\partial(r_{n-1}, x_n)} = \begin{bmatrix} \partial r_n/\partial r_{n-1} & \partial \phi_n/\partial r_{n-1} \\ \partial r_n/\partial x_n & \partial \phi_n/\partial x_n \end{bmatrix} = \begin{bmatrix} \sin\phi_n & r_n^{-1}\cos\phi_n \\ \cos\phi_n & -r_n^{-1}\sin\phi_n \end{bmatrix}.$$

We can illustrate with polar coordinates in case $n = 2$ why this is a better approach than calculating the Jacobian matrix $\partial(r_{n-1}, x_n)/\partial(r_n, \phi_n)$ directly. In this case, equations (2) can be inverted as $r = \sqrt{x^2 + y^2}$ and $\theta = \arctan(y/x)$. The problem is that $\arctan(y/x)$ is not defined if $x = 0$, even if $y \neq 0$, when equations (2) are invertible. In fact, we can also take

$$\theta = \frac{\pi}{4} + \arctan\frac{y - x}{y + x}$$

instead, and there is no trouble with this expression when $x = 0$ and $y \neq 0$. This problem is avoided by obtaining the desired Jacobian as the inverse of another Jacobian.

So, writing $D_{r_n} = \partial/\partial r_n$, $D_{x_n} = \partial/\partial x_n$, $D_{r_{n-1}} = \partial/\partial r_{n-1}$, and $D_{\phi_n} = \partial/\partial \phi_n$, according to the chain rule we have

(3)
$$D_{r_{n-1}} = \frac{\partial r_n}{\partial r_{n-1}}\frac{\partial}{\partial r_n} + \frac{\partial \phi_n}{\partial r_{n-1}}\frac{\partial}{\partial \phi_n} = \sin\phi_n D_{r_n} + \frac{\cos\phi_n}{r_n} D_{\phi_n}$$

and

$$D_{x_n} = \frac{\partial r_n}{\partial x_n}\frac{\partial}{\partial r_n} + \frac{\partial \phi_n}{\partial x_n}\frac{\partial}{\partial \phi_n} = \cos\phi_n D_{r_n} - \frac{\sin\phi_n}{r_n} D_{\phi_n}$$

1.3 Calculating the Laplacian in polar coordinates

We want to evaluate the differential operator $D_{r_{n-1}}^2 + D_{x_n}^2$. In case $n = 2$ we have $r_1 = x_1$ and so this is in fact the Laplacian; in case $n > 2$ the formula we obtain will give a recursive equation that will help in evaluating the Laplacian. To speed up the evaluation of the differential operator $D_{r_{n-1}}^2$ one may observe that the result will consist of a sum second order differential operators, which may be called the principal part, and a sum of first order differential operators. The principal part results from moving all differential operators to the right (as one of the terms in using the product rule of differentiation), and the first order part results when the the differential operator on the left is used up in differentiating a function following it (as the other term in using the product rule). A brief consideration shows that principal part can be calculated by using the binomial theorem; see [1, pp. 3-5] for details All this assumes that these differential operators are applied to functions that the order of derivatives in mixed second derivatives does not make any difference; for this it is sufficient to assume that the functions are twice continuously differentiable, but the conclusion also holds under weaker assumptions; see [2]. We have

$$\begin{aligned} D^2_{r_{n-1}} &= \left(\sin\phi_n D_{r_n} + \frac{\cos\phi_n}{r_n} D_{\phi_n}\right)\left(\sin\phi_n D_{r_n} + \frac{\cos\phi_n}{r_n} D_{\phi_n}\right) \\ &= \sin^2\phi_n D^2_{r_n} + \frac{2\sin\phi_n\cos\phi_n}{r_n} D_{r_n} D_{\phi_n} + \frac{\cos^2\phi_n}{r_n^2} D^2_{\phi_n} \\ &\quad - \frac{\cos\phi_n}{r_n^2} D_{\phi_n} + \frac{\cos^2\phi_n}{r_n} D_{r_n} - \frac{\cos\phi_n\sin\phi_n}{r_n^2} D_{\phi_n}. \end{aligned}$$

Similarly,

$$\begin{aligned} D^2_{x_n} &= \left(\cos\phi_n D_{r_n} - \frac{\sin\phi_n}{r_n} D_{\phi_n}\right)\left(\cos\phi_n D_{r_n} - \frac{\sin\phi_n}{r_n} D_{\phi_n}\right) \\ &= \cos^2\phi_n D^2_{r_n} - \frac{2\cos\phi_n\sin\phi_n}{r_n} D_{r_n} D_{\phi_n} + \frac{\sin^2\phi_n}{r_n^2} D^2_{\phi_n} \\ &\quad + \frac{\cos\phi_n}{r_n^2} D_{\phi_n} + \frac{\sin^2\phi_n}{r_n} D_{r_n} + \frac{\sin\phi_n\cos\phi_n}{r_n^2} D_{\phi_n}. \end{aligned}$$

Adding these two equations, we obtain

(4) $$D^2_{r_{n-1}} + D^2_{x_n} = D^2_{r_n} + \frac{1}{r_n^2} D^2_{\phi_n} + \frac{1}{r_n} D_{r_n}.$$

For $n = 2$, taking have $r_{n-1} = y$, $x_n = x$, $r_n = r = \sqrt{x^2+y^2}$, and $\phi_n = \theta$, this gives the Laplace operator in two dimensions:

(5) $$D^2_x + D^2_y = D^2_r + \frac{1}{r} D_r + \frac{1}{r^2} D^2_\theta = \frac{1}{r} D_r r D_r + \frac{1}{r^2} D^2_\theta.$$

For $n = 3$ with $y = x_1$, $x = x_2$, $z = x_3$ we have $\theta = \phi_2$, $\phi = \phi_3$, $r = r_3$, and $\rho = \sqrt{x^2+y^2} = r_2 = r\sin\phi$. According to (5) we have

$$\begin{aligned} \Delta &= (D^2_x + D^2_y) + D^2_z = \left(D^2_\rho + \frac{1}{\rho} D_\rho + \frac{1}{\rho^2} D_\theta\right) + D^2_z \\ &= \frac{1}{\rho} D_\rho + \frac{1}{\rho^2} D_\theta + (D^2_\rho + D^2_z) = \frac{1}{r\sin\phi} D_\rho + \frac{1}{r^2\sin^2\phi} D^2_\theta + \left(D^2_r + \frac{1}{r} D_r + \frac{1}{r^2} D^2_\phi\right), \end{aligned}$$

where the for the last equation we used (4) and the equation $\rho = \sin\phi$. Using (3) with $n = 3$ to express D_ρ on the right-hand side, we obtain we obtain

$$\begin{aligned}\Delta &= \frac{1}{r\sin\phi}\left(\sin\phi D_r + \frac{\cos\phi}{r}D_\phi\right) + \frac{1}{r^2\sin^2\phi}D_\theta^2 + \left(D_r^2 + \frac{1}{r}D_r + \frac{1}{r^2}D_\phi^2\right),\\ &= D_r^2 + \frac{2}{r}D_r + \frac{\cos\phi}{r^2\sin\phi}D_\phi + \frac{1}{r^2}D_\phi^2 + \frac{1}{r^2\sin^2\phi}D_\theta^2\\ &= \frac{1}{r^2}D_r r^2 D_r + \frac{1}{r^2\sin\phi}D_\phi\sin\phi D_\phi + \frac{1}{r^2\sin^2\phi}D_\theta^2.\end{aligned}$$

As n gets larger, the expression for the Laplacian gets more complicated; in any case, for any $n \ge 2$ we have

$$\sum_{i=1}^{n} D_{x_i}^2 = D_{r_n}^2 + \frac{n-1}{r_n}D_{r_n} + \frac{1}{r_n^2}\Lambda_n, \tag{6}$$

where Λ_n is a second order differential operator involving only the angular variables ϕ_2, ϕ_3, ..., ϕ_n, (i.e., it does not involve r_n). Indeed, assuming that this is true for $n-1$ replacing n, where $n > 2$, and using the equations $r_{n-1} = r_n \sin\phi_n$ and equations (3) and (4), we obtain

$$\begin{aligned}\sum_{i=1}^{n} D_{x_i}^2 &= \sum_{i=1}^{n-1} D_{x_i}^2 + D_{x_n}^2 = \frac{1}{r_{n-1}^2}\Lambda_{n-1} + \frac{n-2}{r_{n-1}}D_{r_{n-1}} + D_{r_{n-1}}^2 + D_{x_n}^2\\ &= \frac{1}{r_n^2\sin^2\phi_n}\Lambda_{n-1} + \frac{n-2}{r_n\sin\phi_n}\left(\sin\phi_n D_{r_n} + \frac{\cos\phi_n}{r_n}D_{\phi_n}\right) + \left(D_{r_n}^2 + \frac{1}{r_n^2}D_{\phi_n}^2 + \frac{1}{r_n}D_{r_n}\right)\\ &= D_{r_n}^2 + \frac{n-1}{r_n}D_{r_n} + \frac{1}{r_n^2}\left(\frac{1}{\sin^2\phi_n}\Lambda_{n-1} + (n-2)\tan\phi_n D_{\phi_n} + D_{\phi_n}^2\right).\end{aligned}$$

This establishes (6) with

$$\Lambda_n = \frac{1}{\sin^2\phi_n}\Lambda_{n-1} + (n-2)\tan\phi_n D_{\phi_n} + D_{\phi_n}^2.$$

Equation (6) can also be written as

$$\sum_{i=1}^{n} D_{x_i}^2 = \frac{1}{r_n^{n-1}}D_{r_n} r_n^{n-1} D_{r_n} + \frac{1}{r_n^2}\Lambda_n. \tag{7}$$

2 Changes of variables in harmonic functions

A function u is called harmonic if it satisfies the Laplace equation $\Delta u = 0$. It is of great interest of finding functions that are harmonic in a region. Once such a function has been found, by a change of variables we may be able to find a harmonic function in a different region.

2.1 Inversion with respect to the unit sphere

We consider polar coordinates in $\mathbb{R}^n$, where $n \ge 2$ is an integer. Instead of r_n we will simply write r. For $(\phi_2, \dots, \phi_n)$ we may write $\boldsymbol{\phi}$, so that the polar coordinates of a point may be written as $(r, \boldsymbol{\phi})$. The change of variables called *inversion* with respect to the unit sphere maps the point $(r, \boldsymbol{\phi})$ to $(1/r, \boldsymbol{\phi})$. Such a change of variables has an important property with respect to harmonic functions, in that, given a harmonic function, using inversion we can find a new harmonic function, as described in the following

Theorem 1. *If $u(r,\boldsymbol{\phi})$ is harmonic at the point $(r^{(0)},\boldsymbol{\phi}^{(0)})$ of $\mathbb{R}^n$, then $r^{2-n}u(1/r,\boldsymbol{\phi})$ is harmonic at the point $(1/r^{(0)},\boldsymbol{\phi}^{(0)})$.*

Proof. Write $\rho=1/r$. Write $\Delta_{(r,\boldsymbol{\phi})}$ for the Laplace operator in polar coordinates $(r,\boldsymbol{\phi})$; see (6) or (7), where now we write r instead of r_n. We will show that

$$\Delta_{(r,\boldsymbol{\phi})}r^{2-n}I=\rho^{n+2}\Delta_{(\rho,\boldsymbol{\phi})}, \tag{8}$$

where I denotes the identity operator. We wrote it on the left-hand side to indicate that we have an operator on the left-hand side, and not the function resulting from applying the operator $\Delta_{(r,\boldsymbol{\phi})}$ to the function whose value of is the expression r^{2-n}; the latter would be written as $\Delta_{(r,\boldsymbol{\phi})}r^{2-n}$ (the point is, that to make sure we indicate an operator, the last symbol of the term should be an operator; the expression on the right-hand side ends with an operator, so there is no need to add I on the right). The assertion of the theorem clearly follows from this equation. Indeed, applying the operators on both sides of equations (8) to the function $u(1/r,\boldsymbol{\phi})=u(\rho,\boldsymbol{\phi})$, we obtain that

$$\Delta_{(r,\boldsymbol{\phi})}r^{2-n}u(1/r,\boldsymbol{\phi})=\rho^{n+2}\Delta_{(\rho,\boldsymbol{\phi})}u(\rho,\boldsymbol{\phi}).$$

If u is harmonic at the point $(r^{(0)},\boldsymbol{\phi}^{(0)})$, the right-hand side is zero for $\rho=r^{(0)}$ and $\boldsymbol{\phi}=\boldsymbol{\phi}^{(0)}$, so the left-hand side must be zero at $r=1/\rho=1/r^{(0)}$ and $\boldsymbol{\phi}=\boldsymbol{\phi}^{(0)}$.

To establish (8), observe that, by the chain rule, we have

$$D_r=\frac{d\rho}{dr}D_\rho=-\frac{1}{r^2}D_\rho=-\rho^2D_\rho. \tag{9}$$

Hence, using (7) with r replacing r_n, we obtain

$$\begin{aligned}\Delta_{(r,\boldsymbol{\phi})}r^{2-n}I&=r^{1-n}D_rr^{n-1}D_rr^{2-n}I+r^{-n}\Lambda_n\\&=r^{1-n}D_rr^{n-1}\big((2-n)r^{1-n}I+r^{2-n}D_r\big)+r^{-n}\Lambda_n,\end{aligned}$$

where the right-hand side was obtained by the product rule; according to this rule $D_rfI=((D_rf)I+fD_r$ for and arbitrary function f. The right-hand side further equals

$$\begin{aligned}r^{1-n}D_r\big((2-n)I+rD_r\big)+r^{-n}\Lambda_n&=r^{1-n}\big((2-n)D_r+D_rrD_r\big)+r^{-n}\Lambda_n\\&=\rho^{n-1}\big((2-n)(-\rho^2)D_\rho+(-\rho^2)D_\rho\rho^{-1}(-\rho^2)D_\rho\big)+\rho^n\Lambda_n\\&=\rho^{n-1}\rho^2\big((n-2)D_\rho+D_\rho\rho D_\rho\big)+\rho^n\Lambda_n,\end{aligned}$$

where the second line was obtained by using (9) and the equation $\rho=r^{-1}$. According to the product rule we have $D_\rho\rho I=I+\rho D_\rho$; hence the right-hand side further equals

$$\begin{aligned}\rho^{n+1}\big((n-2)D_\rho+D_\rho+\rho D_\rho^2\big)+\rho^n\Lambda_n&=\rho^{n+1}\big((n-1)D_\rho+\rho D_\rho^2\big)+\rho^n\Lambda_n\\&=\rho^{n+2}\left(\frac{n-1}{\rho}D_\rho+D_\rho^2+\frac{1}{\rho^2}\Lambda_n\right)=\rho^{n+2}\Delta_{(\rho,\boldsymbol{\phi})},\end{aligned}$$

where the last equation follows according to (6). This establishes (8), completing the proof. □

References

[1] Attila Máté. Characteristic manifolds of linear partial differential equations. http://www.sci.brooklyn.cuny.edu/~mate/misc/partial_diffeqs/character_lin_partdiff_eq.pdf, September 2011.

[2] Attila Máté. On the equality of mixed partial derivatives. http://www.sci.brooklyn.cuny.edu/~mate/misc/mixedpartial.pdf, September 2011.