On the equality of mixed partial derivatives

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According to an example given by Giuseppe Peano, given

\[ f(x, y) = \begin{cases} \frac{xy(x^2 - y^2)}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0), \end{cases} \]

we have \( f_{xy}(0, 0) = -1 \) and \( f_{yx}(0, 0) = 1 \).

To show this, note that, for \( (x, y) \neq (0, 0) \),

\[ f(x, y) = \frac{x^3y - xy^3}{x^2 + y^2}, \]

and so, for \( (x, y) \neq (0, 0) \),

\[ f_x(x, y) = \frac{(3x^2y - y^3)(x^2 + y^2) - (x^3y - xy^3)2x}{(x^2 + y^2)^2}. \]

Taking \( x = 0 \) here, we obtain that

\[ f_x(0, y) = \frac{-y^5}{y^4} = -y \]

for \( y \neq 0 \). Further, for \( y = 0 \) we have \( f(x, y) = 0 \) (whether or not \( x = 0 \)), so \( f_x(x, 0) = 0 \) for any \( x \).

In particular, \( f_x(0, 0) = 0 \). Hence

\[ f_{xy}(0, 0) = \lim_{y \to 0} \frac{f_x(0, y) - f_x(0, 0)}{y} = \lim_{y \to 0} \frac{(-y) - 0}{y} = -1. \]

Similarly, for \( (x, y) \neq (0, 0) \),

\[ f_y(x, y) = \frac{(x^3 - 3xy^2)(x^2 + y^2) - (x^3y - xy^3)2y}{(x^2 + y^2)^2}. \]

Taking \( y = 0 \) here, we obtain that

\[ f_y(x, 0) = \frac{x^5}{x^4} = x \]

for \( y \neq 0 \). Further, for \( x = 0 \) we have \( f(x, y) = 0 \) (whether or not \( y = 0 \)), so \( f_y(0, y) = 0 \) for any \( y \).

In particular, \( f_y(0, 0) = 0 \). Hence

\[ f_{yx}(0, 0) = \lim_{x \to 0} \frac{f_y(x, 0) - f_y(0, 0)}{y} = \lim_{x \to 0} \frac{x - 0}{x} = 1. \]

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However, under reasonable conditions, one can show that \( f_{xy}(x, y) = f_{yx}(x, y) \). The simplest result in this direction is

**Theorem 1.** (A. C. Clairaut) Let \( f \) be a function of two variables, let \((a, b)\) be a point, and let \( U \) be a disk with center \((a, b)\). Assume that \( f \) is defined on \( U \) and its partial derivatives \( f_x, f_y, \) and \( f_{xy} \) and \( f_{yx} \) exist on \( U \). Assume, further, that \( f_{xy} \) and \( f_{yx} \) are continuous at \((a, b)\). Then \( f_{xy}(a, b) = f_{yx}(a, b) \).

**Proof.** Let \((x, y) \in U\) be such that \( x \neq a \) and \( y \neq b \). Applying the Mean-Value Theorem of Differentiation with the function \( f(t, y) - f(t, b) \) as a function of the single variable \( t \), we obtain that there is a \( \xi_0 \) between \( a \) and \( x \) (that is, \( \xi \in (a, x) \) if \( a < x \) and \( \xi_0 \in (x, a) \) if \( x < a \)) such that

\[
f(x, y) - f(x, b) - f(a, y) + f(a, b) = (f(x, y) - f(x, b)) - (f(a, y) - f(a, b)) = (x - a)(f_x(\xi_0, y) - f_x(\xi_0, b)).
\]

Applying the Mean-Value Theorem of Differentiation, now with the function \( f_x(\xi_0, t) \) as a function of the single variable \( t \), we obtain that the right-hand side equals

\[
(y - b)(x - a)f_{xy}(\xi_0, \eta_0)
\]

for some \( \eta_0 \) between \( y \) and \( b \).

Similarly, applying the Mean-Value Theorem of Differentiation with the function \( f(x, t) - f(a, t) \) as a function of the single variable \( t \), we obtain that there is a \( \eta_1 \) between \( b \) and \( y \) such that

\[
f(x, y) - f(x, b) - f(a, y) + f(a, b) = (f(x, y) - f(a, y)) - (f(x, b) - f(a, b)) = (y - b)(f_y(x, \eta_1) - f_y(a, \eta_1)).
\]

Applying the Mean-Value Theorem of Differentiation, now with the function \( f_y(t, \eta_1) \) as a function of the single variable \( t \), we obtain that the right-hand side equals

\[
(x - a)(y - b)f_{yx}(\xi_1, \eta_1)
\]

for some \( \xi_1 \) between \( x \) and \( a \).

Since \( x \neq a \) and \( y \neq b \), it follows that

\[
f_{xy}(\xi_0, \eta_0) = f_{yx}(\xi_1, \eta_1)
\]

for some \( \xi_0 \) and \( \xi_1 \) between \( a \) and \( x \) and for some \( \eta_0 \) and \( \eta_1 \) between \( b \) and \( y \). If \( x \to a \) and \( y \to b \) then \( \xi_0 \to a \), \( \xi_1 \to a \), \( \eta_0 \to b \), and \( \eta_1 \to b \). As \( f_{xy} \) and \( f_{yx} \) are continuous at \((a, b)\), the limit of the left-hand side is \( f_{xy}(a, b) \) and the limit of the right-hand side is \( f_{yx}(a, b) \). These two limits agree, so \( f_{xy}(a, b) = f_{yx}(a, b) \) as we wanted to show. \(\square\)

**Theorem 2.** (H. A. Schwarz) Let \( f \) be a function of two variables, let \((a, b)\) be a point, and let \( U \) be a disk with center \((a, b)\). Assume that \( f \) is defined on \( U \) and its partial derivatives \( f_x, f_y, \) and \( f_{xy} \) exist on \( U \). Assume, further, that \( f_{xy} \) is continuous at \((a, b)\). Then \( f_{yx} \) also exist at \((a, b)\) and \( f_{xy}(a, b) = f_{yx}(a, b) \).

**Proof.** Let \( \epsilon > 0 \) be arbitrary, and let \( \delta > 0 \) be such that \( \delta \) is less than the radius of the disk \( U \) and such that for \((x, y)\) with \( \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta \) we have \( |f_{xy}(x, y) - f_{xy}(a, b)| < \epsilon \). Now, let \((x, y)\) be such that \( x \neq a \), \( y \neq b \), and \( \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta \). Similarly as in the proof of Theorem 1, we have

\[
f(x, y) - f(x, b) - f(a, y) + f(a, b) = (f(x, y) - f(x, b)) - (f(a, y) - f(a, b)) = (y - b)(x - a)f_{xy}(\xi_0, \eta_0)
\]
for some $\xi_0$ between $a$ and $x$ and $\eta_0$ between $b$ and $y$. Thus
\[
\frac{1}{x - a} \left( f(x, y) - f(x, b) \right) - \frac{f(a, y) - f(a, b)}{y - b} = f_{xy}(\xi_0, \eta_0).
\]
As $\sqrt{(\xi - x_0)^2 + (\eta - y_0)^2} < \delta$, we have $f_{xy}(a, b) - \epsilon < f_{xy}(\xi_0, \eta_0) < f_{xy}(a, b) + \epsilon$; hence
\[
f_{xy}(a, b) - \epsilon < \frac{1}{x - a} \left( f(x, y) - f(x, b) \right) - \frac{f(a, y) - f(a, b)}{y - b} < f_{xy}(a, b) + \epsilon.
\]
Making $y \to 0$ here, we obtain that
\[
f_{xy}(a, b) - \epsilon < \frac{1}{x - a} (f_y(x, b) - f_y(a, b)) \leq f_{xy}(a, b) + \epsilon;
\]
note that we used the fact that $f_y$ exists in $U$, so the limits of the difference quotients above do exist. Since for every $\epsilon > 0$ there is $\delta$ such that for all $x$ with $0 < |x - a| < \delta$ the last inequality holds, this means that
\[
\lim_{x \to a} \frac{f_y(x, b) - f_y(a, b)}{x - a} = f_{xy}(a, b).
\]
This is what we wanted to show, since the limit on the left-hand side is $f_{yx}(a, b)$. \hfill \Box

**Theorem 3.** (W. H. Young) Let $f$ be a function of two variables, let $(a, b)$ be a point, and let $U$ be a disk with center $(a, b)$. Assume that $f$ is defined on $U$ and its partial derivatives $f_x, f_y$ exist on $U$. Assume, further, that $f_x$ and $f_y$ are totally differentiable at $(a, b)$. Then $f_{xy}(a, b) = f_{yx}(a, b)$.

**Proof.** The total differentiability of $f_x$ at $(a, b)$ means that for $(x, y) \in U$ we have
\[
(1) \quad f_x(x, y) = f_x(a, b) + f_{xx}(a, b)(x - a) + f_{xy}(a, b)(y - b) + \epsilon_1(x, y)(x - a) + \epsilon_2(x, y)(y - b),
\]
where $\epsilon_1$ and $\epsilon_2$ are functions on $U$ such that
\[
(2) \quad \lim_{(x, y) \to (a, b)} \epsilon_1(x, y) = 0 \quad \text{and} \quad \lim_{(x, y) \to (a, b)} \epsilon_2(x, y) = 0.
\]
Similarly, the total differentiability of $f_y$ at $(a, b)$ means that for $(x, y) \in U$ we have
\[
(3) \quad f_y(x, y) = f_y(a, b) + f_{yx}(a, b)(x - a) + f_{yy}(a, b)(y - b) + \epsilon_3(x, y)(x - a) + \epsilon_4(x, y)(y - b),
\]
where $\epsilon_3$ and $\epsilon_4$ are functions on $U$ such that
\[
(4) \quad \lim_{(x, y) \to (a, b)} \epsilon_3(x, y) = 0 \quad \text{and} \quad \lim_{(x, y) \to (a, b)} \epsilon_4(x, y) = 0.
\]

One might be tempted to use equations (1) and (3) to evaluate $f(x, y)$ in two different ways:
\[
f(x, y) = f(a, b) + (f(x, b) - f(a, b)) + (f(x, y) - f(x, b))
= f(a, b) + \int_a^x f_x(t, b) \, dt + \int_y^y f_y(x, t) \, dt,
\]
and
\[
f(x, y) = f(a, b) + (f(a, y) - f(a, b)) + (f(x, y) - f(a, y))
= f(a, b) + \int_b^y f_y(a, t) \, dt + \int_a^a f_x(t, y) \, dt.
\]
The first one of these will involve $f_{yy}(a, b)$ and the second one, $f_{xy}(a, b)$, so by comparing these two expressions for $f(x, y)$ one should be able to obtain the equality of these mixed partial derivatives. Unfortunately, our assumptions are not strong enough to guarantee the existence of the integrals on the right-hand side. Therefore, we have to proceed in a somewhat roundabout way, using the Mean-Value Theorem of Differentiation instead of the above integral formulas. Thus, the above formulas will be replaced by formulas (6) and (8) below. These latter formulas will serve much the same function as the integral formulas above, and they are valid under the present assumptions.

For $(x, y) \in U$ write

$$F(x, y) = f(x, y) - \left( f_x(a, b)(x - a) + f_y(a, b)(y - b) + f_{xx}(a, b)\frac{(x - a)^2}{2} + f_{yx}(a, b)(x - a)(y - b) + f_{yy}(a, b)\frac{(y - b)^2}{2} \right).$$

(5)

Note that we have $F_x(x, b) = f_x(x, b) - (f_y(a, b) + f_{yx}(a, b)(x - a)) = \epsilon_1(x, b)(x - a)$ in view of (1), and

$$F_y(x, y) = f_y(x, y) - (f_y(a, b) + f_{yx}(a, b)(x - a) + f_{yy}(y - b)) = \epsilon_3(x, y)(x - a) + \epsilon_4(x, y)(y - b)$$

in view of (3). Hence, assuming that $x \neq a$ and $y \neq b$, by using the Mean-Value Theorem of differentiation twice, we obtain

$$F(x, y) - F(a, b) = (F(x, b) - F(a, b)) + (F(x, y) - F(x, b))$$

(6)

$$= F_x(\xi_1, b)(x - a) + F_y(x, \eta_1)(y - b)$$

$$= \epsilon_1(\xi_1, b)(x - a) + \epsilon_3(x, \eta_1)(x - a)^2 + \epsilon_4(x, \eta_1)(\eta_1 - b)(y - b)$$

for some $\xi_1$ between $a$ and $x$ (i.e., $a < \xi_1 < x$ if $a < x$ and $x < \xi_1 < a$ if $x < a$) and $\eta_1$ between $b$ and $y$.

Similarly, for $(x, y) \in U$ write

$$G(x, y) = f(x, y) - \left( f_x(a, b)(x - a) + f_y(a, b)(y - b) + f_{xx}(a, b)\frac{(x - a)^2}{2} + f_{yx}(a, b)(x - a)(y - b) + f_{yy}(a, b)\frac{(y - b)^2}{2} \right).$$

(7)

(the only difference from the definition of $F(x, y)$ is that here we take $f_{xy}$, whereas before we took $f_{yx}$). Note that we have

$$G_x(x, y) = f_x(x, y) - (f_x(a, b) + f_{xx}(a, b)(x - a) + f_{xy}(a, b)(y - b)) = \epsilon_1(x, y)(x - a) + \epsilon_2(x, y)(y - b)$$

in view of (1), and

$$G_y(a, y) = f_y(a, y) - (f_y(a, b) + f_{yy}(a, b)(y - b)) = \epsilon_4(a, y)(y - b)$$
in view of (3). Hence, assuming that \( x \neq a \) and \( y \neq b \), by using the Mean-Value Theorem of differentiation twice, we obtain

\[
G(x, y) - G(a, b) = (G(x, y) - G(a, y)) + (G(a, y) - G(a, b))
\]

\[
= G_x(\xi_2, y)(x - a) + G_y(a, \eta_2)(y - b)
\]

\[
= \epsilon_1(\xi_2, y)(\xi_2 - a)(x - a) + \epsilon_2(\xi_2, y)(y - b)(x - a) + \epsilon_4(a, \eta_2)(\eta_2 - b)(y - b)
\]

for some \( \xi_2 \) between \( a \) and \( x \) and \( \eta_2 \) between \( b \) and \( y \).

By (5) and (7) we have

\[
F(x, y) - G(x, y) = (f_{xy}(a, b) - f_{yx}(a, b))(x - a)(y - b).
\]

Further, also by by (5) and (7) we have \( f(a, b) = F(a, b) = G(a, b) \). Hence, by (6) and (8) we can see that

\[
F(x, y) - G(x, y) = \epsilon_1(\xi_1, b)(\xi_1 - a)(x - a) + \epsilon_3(x, \eta_1)(x - a)^2 + \epsilon_4(x, \eta_1)(\eta_1 - b)(y - b)
\]

\[
- \epsilon_1(\xi_2, y)(\xi_2 - a)(x - a) - \epsilon_2(\xi_2, y)(y - b)(x - a) - \epsilon_4(a, \eta_2)(\eta_2 - b)(y - b)
\]

The right-hand sides of equations (9) and (10) must be equal. To simplify the equation so obtained, assume that \( x = a + h \) and \( y = b + h \) for some \( h > 0 \) (but \( h \) is small enough so that \( (x, y) \in U \), as required for the above equations to hold). Note that then \( 0 < \xi_1 - a < h, 0 < \xi_2 - a < h, 0 < \eta_1 - b < h, \) and \( 0 < \eta_2 - b < h \). Writing \( \theta_1 = (\xi_1 - a)/h, \theta_2 = (\xi_2 - a)/h, \theta_3 = (\eta_1 - b)/h, \) and \( \theta_4 = (\eta_2 - b)/h \), equating the right-hand sides of (9) and (10), and dividing through by \( h^2 \), we obtain that

\[
f_{xy}(a, b) - f_{yx}(a, b) = \epsilon_1(\xi_1, b)\theta_1 + \epsilon_3(x, \eta_1) + \epsilon_4(x, \eta_1)\theta_3
\]

\[
- \epsilon_1(\xi_2, y)\theta_2 - \epsilon_2(\xi_2, y) - \epsilon_4(a, \eta_2)\theta_4,
\]

where \( 0 < \theta_i < 1 \) for \( i = 1, 2, 3, 4 \). Making \( h \to 0 \), we have \( x \to a, y \to b, \xi_1 \to a, \xi_2 \to a, \eta_1 \to b, \) and \( \eta_2 \to b \). Hence the limit of the right-hand side is 0 in view of (2) and (4). So the left-hand side is also 0, i.e., \( f_{xy}(a, b) = f_{yx}(a, b) \), as we wanted to show. \( \square \)