On the equality of mixed partial derivatives*

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According to on example given by Giuseppe Peano, given

$$f(x,y)=\begin{cases}\dfrac{xy(x^2-y^2)}{x^2+y^2} & \text{if}\quad (x,y)\ne(0,0),\\ 0 & \text{if}\quad (x,y)=(0,0),\end{cases}$$

we have $f_{xy}(0,0)=-1$ and $f_{yx}(0,0)=1$.

To show this, note that, for $(x,y)\ne(0,0)$,

$$f(x,y)=\frac{x^3y-xy^3}{x^2+y^2},$$

and so, for $(x,y)\ne(0,0)$,

$$f_x(x,y)=\frac{(3x^2y-y^3)(x^2+y^2)-(x^3y-xy^3)2x}{(x^2+y^2)^2}.$$

Taking $x=0$ here, we obtain that

$$f_x(0,y)=\frac{-y^5}{y^4}=-y$$

for $y\ne0$. Further, for $y=0$ we have $f(x,y)=0$ (whether or not $x=0$), so $f_x(x,0)=0$ for any x. In particular, $f_x(0,0)=0$. Hence

$$f_{xy}(0,0)=\lim_{y\to0}\frac{f_x(0,y)-f_x(0,0)}{y}=\lim_{y\to0}\frac{(-y)-0}{y}=-1.$$

Similarly, for $(x,y)\ne(0,0)$,

$$f_y(x,y)=\frac{(x^3-3xy^2)(x^2+y^2)-(x^3y-xy^3)2y}{(x^2+y^2)^2}.$$

Taking $y=0$ here, we obtain that

$$f_y(x,0)=\frac{x^5}{x^4}=x$$

for $y\ne0$. Further, for $x=0$ we have $f(x,y)=0$ (whether or not $y=0$), so $f_y(0,y)=0$ for any y. In particular, $f_y(0,0)=0$. Hence

$$f_{yx}(0,0)=\lim_{x\to0}\frac{f_y(x,0)-f_y(0,0)}{y}=\lim_{x\to0}\frac{x-0}{x}=1.$$

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However, under reasonable conditions, one can show that $f_{xy}(x,y) = f_{yx}(x,y)$. The simplest result in this direction is

Theorem 1. (A. C. Clairaut) *Let f be a function of two variables, let (a,b) be a point, and let U be a disk with center (a,b). Assume that f is defined on U and its partial derivatives f_x, f_y, and f_{xy} and f_{yx} exist on U. Assume, further, that f_{xy} and f_{yx} are continuous at (a,b). Then $f_{xy}(a,b) = f_{yx}(a,b)$.*

Proof. Let $(x,y) \in U$ be such that $x \neq a$ and $y \neq b$. Applying the Mean-Value Theorem of Differentiation with the function $f(t,y) - f(t,b)$ as a function of the single variable t, we obtain that there is a ξ_0 between a and x (that is, $\xi \in (a,x)$ if $a < x$ and $\xi_0 \in (x,a)$ if $x < a$) such that

$$\begin{aligned} &f(x,y) - f(x,b) - f(a,y) + f(a,b) = (f(x,y) - f(x,b)) - (f(a,y) - f(a,b)) \\ &\quad = (x-a)(f_x(\xi_0,y) - f_x(\xi_0,b)). \end{aligned}$$

Applying the Mean-Value Theorem of Differentiation, now with the function $f_x(\xi_0,t)$ as a function of the single variable t, we obtain that the right-hand side equals

$$(y-b)(x-a)f_{xy}(\xi_0,\eta_0)$$

for some η_0 between y and b.

Similarly, applying the Mean-Value Theorem of Differentiation with the function $f(x,t) - f(a,t)$ as a function of the single variable t, we obtain that there is a η_1 between b and y such that

$$\begin{aligned} &f(x,y) - f(x,b) - f(a,y) + f(a,b) = (f(x,y) - f(a,y)) - (f(x,b) - f(a,b)) \\ &\quad = (y-b)(f_y(x,\eta_1) - f_y(a,\eta_1)); \end{aligned}$$

Applying the Mean-Value Theorem of Differentiation, now with the function $f_y(t,\eta_1)$ as a function of the single variable t, we obtain that the right-hand side equals

$$(x-a)(y-b)f_{yx}(\xi_1,\eta_1)$$

for some ξ_1 between x and a.

Since $x \neq a$ and $y \neq b$, it follows that

$$f_{xy}(\xi_0,\eta_0) = f_{yx}(\xi_1,\eta_1)$$

for some ξ_0 and ξ_1 between a and x and for some η_0 and η_1 between b and y. If $x \to a$ and $y \to b$ then $\xi_0 \to a$, $\xi_1 \to a$, $\eta_0 \to b$, and $\eta_1 \to b$. As f_{xy} and f_{yx} are continuous at (a,b), the limit of the left-hand side is $f_{xy}(a,b)$ and the limit of the right-hand side is $f_{yx}(a,b)$. These two limits agree, so $f_{xy}(a,b) = f_{yx}(a,b)$ as we wanted to show. $\square$

Theorem 2. (H. A. Schwarz) *Let f be a function of two variables, let (a,b) be a point, and let U be a disk with center (a,b). Assume that f is defined on U and its partial derivatives f_x, f_y, and f_{xy} exist on U. Assume, further, that f_{xy} is continuous at (a,b). Then f_{yx} also exist at (a,b) and $f_{xy}(a,b) = f_{yx}(a,b)$.*

Proof. Let $\epsilon > 0$ be arbitrary, and let $\delta > 0$ be such that δ is less than the radius of the disk U and such that for (x,y) with $\sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$ we have $|f_{xy}(x,y) - f_{xy}(a,b)| < \epsilon$. Now, let (x,y) be such that $x \neq a$, $y \neq b$, and $\sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$. Similarly as in the proof of Theorem 1, we have

$$\begin{aligned} &f(x,y) - f(x,b) - f(a,y) + f(a,b) = (f(x,y) - f(x,b)) - (f(a,y) - f(a,b)) \\ &\quad = (y-b)(x-a)f_{xy}(\xi_0,\eta_0) \end{aligned}$$

for some ξ_0 between a and x and η_0 between b and y. Thus

$$\frac{1}{x-a}\left(\frac{f(x,y)-f(x,b)}{y-b}-\frac{f(a,y)-f(a,b)}{y-b}\right)=f_{xy}(\xi_0,\eta_0).$$

As $\sqrt{(\xi-x_0)^2+(\eta_0-y_0)^2}<\delta$, we have $f_{xy}(a,b)-\epsilon<f_{xy}(\xi_0,\eta_0)<f_{xy}(a,b)+\epsilon$; hence

$$f_{xy}(a,b)-\epsilon<\frac{1}{x-a}\left(\frac{f(x,y)-f(x,b)}{y-b}-\frac{f(a,y)-f(a,b)}{y-b}\right)<f_{xy}(a,b)+\epsilon.$$

Making $y\to 0$ here, we obtain that

$$f_{xy}(a,b)-\epsilon\le\frac{1}{x-a}\left(f_y(x,b)-f_y(a,b)\right)\le f_{xy}(a,b)+\epsilon;$$

note that we used the fact that f_y exists in U, so the limits of the difference quotients above do exist. Since for every $\epsilon>0$ there is δ such that for all x with $0<|x-a|<\delta$ the last inequality holds, this means that

$$\lim_{x\to a}\frac{f_y(x,b)-f_y(a,b)}{x-a}=f_{xy}(a,b).$$

This is what we wanted to show, since the limit on the left-hand side is $f_{yx}(a,b)$. □

Theorem 3. (W. H. Young) *Let f be a function of two variables, let (a,b) be a point, and let U be a disk with center (a,b). Assume that f is defined on U and its partial derivatives f_x, f_y exist on U. Assume, further, that f_x and f_y are totally differentiable at (a,b). Then $f_{xy}(a,b)=f_{yx}(a,b)$.*

Proof. The total differentiability of f_x at (a,b) means that for $(x,y)\in U$ we have

$$f_x(x,y)=f_x(a,b)+f_{xx}(a,b)(x-a)+f_{xy}(a,b)(y-b)+\epsilon_1(x,y)(x-a)+\epsilon_2(x,y)(y-b), \tag{1}$$

where ϵ_1 and ϵ_2 are functions on U such that

$$\lim_{(x,y)\to(a,b)}\epsilon_1(x,y)=0\quad\text{and}\quad\lim_{(x,y)\to(a,b)}\epsilon_2(x,y)=0. \tag{2}$$

Similarly, the total differentiability of f_y at (a,b) means that for $(x,y)\in U$ we have

$$f_y(x,y)=f_y(a,b)+f_{yx}(a,b)(x-a)+f_{yy}(a,b)(y-b)+\epsilon_3(x,y)(x-a)+\epsilon_4(x,y)(y-b), \tag{3}$$

where ϵ_3 and ϵ_4 are functions on U such that

$$\lim_{(x,y)\to(a,b)}\epsilon_3(x,y)=0\quad\text{and}\quad\lim_{(x,y)\to(a,b)}\epsilon_4(x,y)=0. \tag{4}$$

One might be tempted to use equations (1) and (3) to evaluate $f(x,y)$ in two different ways:

$$\begin{aligned}f(x,y)&=f(a,b)+\big(f(x,b)-f(a,b)\big)+\big(f(x,y)-f(x,b)\big)\\&=f(a,b)+\int_a^x f_x(t,b)\,dt+\int_b^y f_y(x,t)\,dt,\end{aligned}$$

and

$$\begin{aligned}f(x,y)&=f(a,b)+\big(f(a,y)-f(a,b)\big)+\big(f(x,y)-f(a,y)\big)\\&=f(a,b)+\int_b^y f_y(a,t)\,dt+\int_a^x f_x(t,y)\,dt.\end{aligned}$$

The first one of these will involve $f_{yx}(a,b)$ and the second one, $f_{xy}(a,b)$, so by comparing these two expressions for $f(x,y)$ one should be able to obtain the equality of these mixed partial derivatives. Unfortunately, our assumptions are not strong enough to guarantee the validity of these equations – they are not even strong enough to guarantee the existence of the integrals on the right-hand side. Therefore, we have to proceed in a somewhat roundabout way, using the Mean-Value Theorem of Differentiation instead of the above integral formulas. Thus, the above formulas will be replaced by formulas (6) and (8) below. These latter formulas will serve much the same function as the integral formulas above, and they are valid under the present assumptions.

For $(x,y) \in U$ write

$$
\begin{aligned}
F(x,y) = f(x,y) - \Bigg(& f_x(a,b)(x-a) + f_y(a,b)(y-b) + f_{xx}(a,b)\frac{(x-a)^2}{2} \\
&+ f_{yx}(a,b)(x-a)(y-b) + f_{yy}(a,b)\frac{(y-b)^2}{2} \Bigg).
\end{aligned} \tag{5}
$$

Note that we have

$$F_x(x,b) = f_x(x,b) - (f_x(a,b) + f_{xx}(a,b)(x-a)) = \epsilon_1(x,b)(x-a)$$

in view of (1), and

$$F_y(x,y) = f_y(x,y) - (f_y(a,b) + f_{yx}(x-a) + f_{yy}(y-b)) = \epsilon_3(x,y)(x-a) + \epsilon_4(x,y)(y-b)$$

in view of (3). Hence, assuming that $x \neq a$ and $y \neq b$, by using the Mean-Value Theorem of differentiation twice, we obtain

$$
\begin{aligned}
F(x,y) - F(a,b) &= (F(x,b) - F(a,b)) + (F(x,y) - F(x,b)) \\
&= F_x(\xi_1,b)(x-a) + F_y(x,\eta_1)(y-b) \\
&= \epsilon_1(\xi_1,b)(\xi_1-a)(x-a) + \epsilon_3(x,\eta_1)(x-a)^2 + \epsilon_4(x,\eta_1)(\eta_1-b)(y-b)
\end{aligned} \tag{6}
$$

for some ξ_1 between a and x (i.e., $a < \xi_1 < x$ if $a < x$ and $x < \xi_1 < a$ if $x < a$) and η_1 between b and y.

Similarly, for $(x,y) \in U$ write

$$
\begin{aligned}
G(x,y) = f(x,y) - \Bigg(& f_x(a,b)(x-a) + f_y(a,b)(y-b) + f_{xx}(a,b)\frac{(x-a)^2}{2} \\
&+ f_{xy}(a,b)(x-a)(y-b) + f_{yy}(a,b)\frac{(y-b)^2}{2} \Bigg)
\end{aligned} \tag{7}
$$

(the only difference from the definition of $F(x,y)$ is that here we take f_{xy}, whereas before we took f_{yx}). Note that we have

$$G_x(x,y) = f_x(x,y) - (f_x(a,b) + f_{xx}(a,b)(x-a) + f_{xy}(a,b)(y-b)) = \epsilon_1(x,y)(x-a) + \epsilon_2(x,y)(y-b)$$

in view of (1), and

$$G_y(a,y) = f_y(a,y) - (f_y(a,b) + f_{yy}(a,b)(y-b)) = \epsilon_4(a,y)(y-b)$$

in view of (3). Hence, assuming that $x \neq a$ and $y \neq b$, by using the Mean-Value Theorem of differentiation twice, we obtain

$$
\begin{aligned}
G(x,y) - G(a,b) &= (G(x,y) - G(a,y)) + (G(a,y) - G(a,b)) \\
&= G_x(\xi_2, y)(x-a) + G_y(a,\eta_2)(y-b) \\
&= \epsilon_1(\xi_2,y)(\xi_2 - a)(x-a) + \epsilon_2(\xi_2,y)(y-b)(x-a) + \epsilon_4(a,\eta_2)(\eta_2 - b)(y-b)
\end{aligned}
\tag{8}
$$

for some ξ_2 between a and x and η_2 between b and y.

By (5) and (7) we have

$$
F(x,y) - G(x,y) = \big(f_{xy}(a,b) - f_{yx}(a,b)\big)(x-a)(y-b). \tag{9}
$$

Further, also by by (5) and (7) we have $f(a,b) = F(a,b) = G(a,b)$. Hence, by (6) and (8) we can see that

$$
\begin{aligned}
F(x,y) - G(x,y) &= \epsilon_1(\xi_1,b)(\xi_1 - a)(x-a) + \epsilon_3(x,\eta_1)(x-a)^2 + \epsilon_4(x,\eta_1)(\eta_1 - b)(y-b) \\
&\quad - \epsilon_1(\xi_2,y)(\xi_2 - a)(x-a) - \epsilon_2(\xi_2,y)(y-b)(x-a) - \epsilon_4(a,\eta_2)(\eta_2 - b)(y-b).
\end{aligned}
\tag{10}
$$

The right-hand sides of equations (9) and (10) must be equal. To simplify the equation so obtained, assume that $x = a + h$ and $y = b + h$ for some $h > 0$ (but h is small enough so that $(x,y) \in U$, as required for the above equations to hold). Note that then $0 < \xi_1 - a < h$, $0 < \xi_2 - a < h$, $0 < \eta_1 - b < h$, and $0 < \eta_2 - b < h$. Writing $\theta_1 = (\xi_1 - a)/h$, $\theta_2 = (\xi_2 - a)/h$, $\theta_3 = (\eta_1 - b)/h$, and $\theta_4 = (\eta_2 - b)/h$, equating the right-hand sides of (9) and (10), and dividing through by h^2, we obtain that

$$
\begin{aligned}
f_{xy}(a,b) - f_{yx}(a,b) &= \epsilon_1(\xi_1,b)\theta_1 + \epsilon_3(x,\eta_1) + \epsilon_4(x,\eta_1)\theta_3 \\
&\quad - \epsilon_1(\xi_2,y)\theta_2 - \epsilon_2(\xi_2,y) - \epsilon_4(a,\eta_2)\theta_4,
\end{aligned}
$$

where $0 < \theta_i < 1$ for $i = 1,2,3,4$. Making $h \to 0$, we have $x \to a$, $y \to b$, $\xi_1 \to a$, $\xi_2 \to a$, $\eta_1 \to b$, and $\eta_2 \to b$. Hence the limit of the right-hand side is 0 in view of (2) and (4). So the left-hand side is also 0, i.e., $f_{xy}(a,b) = f_{yx}(a,b)$, as we wanted to show. □