The limit $\lim_{t\to 0} \sin t/t$ *

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Consider the unit circle $x^2 + y^2 = 1$ in the plane. Writing $|AB|$ for the length of the line segment between the point A and B, the length of an arc $\widehat{PQ}$ on the unit circle is defined as the supremum of the sums

$$\sum_{i=1}^{n} |P_{i-1}P_i|$$

for all integers $n > 1$ and all sequences of points along the arc $\widehat{PQ}$ such that $P_0 = P$, $P_n = Q$, and P_i is between P_{i-1} and P_{i+1} for all i with $0 < i < n$. [1] If $A(x_1, y_1)$ and $B(x_2, y_2)$ are distinct points in the first quadrant on the unit circle, then it is easy to see that

$$|x_2 - x_1| < |AB| < |x_2 - x_1| + |y_2 - y_1|. \tag{1}$$

Hence, if the arc $\widehat{PQ}$ lies in the first quadrant, it is easy to see that all the sums above are less than 2, and so the supremum of these sums exists and is ≤ 2. Denoting by π the arc length of the unit semicircle, it follows that $\pi/2 \le 2$. From the equilateral triangle, one can conclude that $\pi/3 > 1$; hence one obtains the simple estimate $3 < \pi \le 4$.

Assuming $P(x_P, y_P)$ and $Q(x_Q, y_Q)$ are points in on the unit circle in the first quadrant such that $x_P > x_Q$ (and then $y_P < y_Q$), and $P_i(x_i, y_i)$ for $0 \le i \le n$ are points along the arc between P and Q as above, we have $x_0 > x_1 > \ldots > x_n$ and $y_0 < y_1 \ldots < y_n$. Furthermore,

$$\begin{aligned} x_P - x_Q = x_0 - x_n = \sum_{i=1}^{n} (x_{i-1} - x_i) < \sum_{i=1}^{n} |P_{i-1}P_i| \\ < \sum_{i=1}^{n} \big((x_{i-1} - x_i) + (y_i - y_1)\big) = (x_0 - x_n) + (y_n - y_0) = (x_P - x_Q) + (y_P - y_Q). \end{aligned}$$

Taking the supremum of the second sum for all choices of n and for all choices of the points P_i, this supremum being the length of the arc $\widehat{PQ}$, we obtain the inequality

$$x_P - x_Q < \widehat{PQ} \le (x_P - x_Q) + (y_P - y_Q). \tag{2}$$

If one writes P_0 for the point with coordinates $(0, 1)$ and denotes by P_t the point (x, y) on the unit circle such that the length of the arc $\widehat{P_0P_t}$ is $|t|$ and the rotation about the origin from P_0 to P_t

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[1] To make it easy to define what is meant by P_i being between P_{i-1} and P_{i+1}, one may add the additional requirement that $|P_{k-1}P_k| < 1/2$ for all k with $0 < k \le n$; in this case, all three points P_{i-1}, P_i, P_{i+1} lie in the same one of at least one of the four half planes $x \ge 0$, $x \le 0$, $y \ge 0$, and $y \le 0$, and then it is easy to define the concept of one point being between the other two.

is counterclockwise for $t \geq 0$ and clockwise for $t \leq 0$, then t is a continuous function of x (or y) for P_t in each quadrant. From here it is easy to conclude with the aid of the Intermediate Value Theorem that to each real t there corresponds a point P_t on the unit circle. With the aid of the point $P_t(x, y)$ one can define the trigonometric functions as $\cos t = x$, $\sin t = y$, and $\tan t = \sin t / \cos t$ (this last one if $\cos t \neq 0$).

Assuming $0 < t < \pi/2$ (so that P_t is in the first quadrant), inequality (2) for the arc $\widehat{P_0 P_t}$ can be written as

$$\sin t = \sin t - \sin 0 < t \leq (\sin t - \sin 0) + (\cos 0 - \cos t) = \sin t + 1 - \cos t.$$

Rearranging this, we obtain

$$1 - \frac{1 - \cos t}{t} \leq \frac{\sin t}{t} < 1. \tag{3}$$

Using the second inequality here and also noting that $0 < \cos t < 1$, we can see that

$$\frac{1 - \cos t}{t} < \frac{(1 - \cos t)(1 + \cos t)}{t} = \frac{1 - \cos^2 t}{t} = t \cdot \frac{\sin^2 t}{t^2} < t.$$

Thus, (3) implies

$$1 - t < \frac{\sin t}{t} < 1.$$

This implies that $\lim_{t \searrow 0} \sin t / t = 1$. As $\sin t / t$ is an even function of t, we can conclude that

$$\lim_{t \to 0} \frac{\sin t}{t} = 1.$$