Telescoping sums*

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1 Some telescoping sums

The following two equations are easy to show. Let a_k be a real number for all integers $k \ge 0$. Then

$$\sum_{k=1}^{n}(a_k - a_{k-1}) = a_n - a_0 \tag{1}$$

for every integer $n \ge 0$. Similarly, let a_k be a real number for all integers $k \ge 1$. Then

$$\sum_{k=1}^{n}(a_k - a_{k+1}) = a_1 - a_{n+1} \tag{2}$$

for every integer $n \ge 0$. Note that for $n \ge 1$ the equations are true since all terms not shown on the right-hand side cancel; for $n = 0$ they are true since the sums on the left are empty, and empty sums are 0 by convention, while the right-hand sides are also 0. Such sums are called telescoping or collapsing sums. The reason for the name "telescoping" is that some telescopes are constructed of several tubes of different diameters that can be collapsed or pushed into one another for compact storing of the telescope.

1.1 An example for formula (1)

As an example for applications of the first equation, consider

$$a_k = \prod_{j=0}^{l}(k + j) \qquad (l \ge 0). \tag{3}$$

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Then, for $k \geq 1$ we have

$$\begin{aligned} a_k - a_{k-1} = \prod_{j=0}^{l}(k+j) - \prod_{j=0}^{l}(k-1+j) &= (k+l)\prod_{j=0}^{l-1}(k+j) - (k-1)\prod_{j=1}^{l}(k-1+j) \\ = (k+l)\prod_{j=0}^{l-1}(k+j) - (k-1)\prod_{j=0}^{l-1}(k+j) &= \big((k+l)-(k-1)\big)\prod_{j=0}^{l-1}(k+j) \\ = (l+1)\prod_{j=0}^{l-1}(k+j). \end{aligned}$$

Observe that this calculation is valid even in case $l = 0$, since the empty product is taken to equal 1 by convention. Noting that $a_0 = 0$, it follows from (1) that

$$\sum_{k=1}^{n}(l+1)\prod_{j=0}^{l-1}(k+j) = \prod_{j=0}^{l}(n+j);$$

that is, dividing both sides by $l + 1$, we obtain

$$\sum_{k=1}^{n}\prod_{j=0}^{l-1}(k+j) = \frac{1}{l+1}\prod_{j=0}^{l}(n+j). \tag{4}$$

For $l = 0$ this gives

$$\sum_{k=1}^{n} 1 = n.$$

For $l = 1$ it gives

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}.$$

For $l = 2$ it gives

$$\sum_{k=1}^{n} k(k+1) = \frac{n(n+1)(n+2)}{3}.$$

For $l = 3$ it gives

$$\sum_{k=1}^{n} k(k+1)(k+2) = \frac{n(n+1)(n+2)(n+3)}{4}.$$

As an application of these formulas, we have

$$\begin{aligned} \sum_{k=1}^{n} k^2 = \sum_{k=1}^{n}\big(k(k+1)-k\big) = \sum_{k=1}^{n}k(k+1) - \sum_{k=1}^{n}k &= \frac{n(n+1)(n+2)}{3} - \frac{n(n+1)}{2} \\ = \frac{2n(n+1)(n+2)-3n(n+1)}{6} = \frac{n(n+1)\big(2(n+2)-3\big)}{6} &= \frac{n(n+1)(2n+1)}{6}. \end{aligned}$$

1.2 An example for formula (2)

As an example for the application of the second equation, consider

$$a_k = \frac{1}{\prod_{j=0}^{l}(k+j)} \qquad (l \geq 0). \tag{5}$$

Then, for $k \geq 1$ we have

$$\begin{aligned}
a_k - a_{k+1} &= \frac{1}{\prod_{j=0}^{l}(k+j)} - \frac{1}{\prod_{j=0}^{l}(k+1+j)} = \frac{1}{\prod_{j=0}^{l}(k+j)} - \frac{1}{\prod_{j=1}^{l+1}(k+j)} \\
&= \frac{k+l+1}{(k+l+1)\prod_{j=0}^{l}(k+j)} - \frac{k}{k\prod_{j=1}^{l+1}(k+j)} = \frac{k+l+1}{\prod_{j=0}^{l+1}(k+j)} - \frac{k}{\prod_{j=0}^{l+1}(k+j)} \\
&= \frac{(k+l+1)-k}{\prod_{j=0}^{l+1}(k+j)} = \frac{l+1}{\prod_{j=0}^{l+1}(k+j)}.
\end{aligned}$$

Thus, by equation (2) we have

$$\begin{aligned}
\sum_{k=1}^{n} \frac{l+1}{\prod_{j=0}^{l+1}(k+j)} &= \frac{1}{\prod_{j=0}^{l}(1+j)} - \frac{1}{\prod_{j=0}^{l}(n+1+j)} \\
&= \frac{1}{\prod_{j=1}^{l+1} j} - \frac{1}{\prod_{j=1}^{l+1}(n+j)} = \frac{1}{(l+1)!} - \frac{1}{\prod_{j=1}^{l+1}(n+j)}
\end{aligned}$$

Dividing both sides by $l+1$, we obtain

$$\sum_{k=1}^{n} \frac{1}{\prod_{j=0}^{l+1}(k+j)} = \frac{1}{l+1}\left(\frac{1}{(l+1)!} - \frac{1}{\prod_{j=1}^{l+1}(n+j)}\right) \tag{6}$$

For $l = 0$ this gives

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = 1 - \frac{1}{n+1}.$$

For $l = 1$ it gives

$$\sum_{k=1}^{n} \frac{1}{k(k+1)(k+2)} = \frac{1}{2}\left(\frac{1}{2} - \frac{1}{(n+1)(n+2)}\right).$$

For $l = 2$ it gives

$$\sum_{k=1}^{n} \frac{1}{k(k+1)(k+2)(k+3)} = \frac{1}{3}\left(\frac{1}{6} - \frac{1}{(n+1)(n+2)(n+3)}\right).$$

For $l = 3$ it gives

$$\sum_{k=1}^{n} \frac{1}{k(k+1)(k+2)(k+3)(k+4)} = \frac{1}{4}\left(\frac{1}{24} - \frac{1}{(n+1)(n+2)(n+3)(n+4)}\right).$$

1.3 Telescoping sums of binomial coefficients

The binomial coefficient $\binom{\alpha}{k}$ for any integer $k \geq 0$ and any real α is defined as

$$\binom{\alpha}{k} \overset{def}{=} \prod_{j=0}^{k-1} \frac{\alpha - j}{k - j} = \frac{1}{k!} \prod_{j=0}^{k-1} (\alpha - j). \tag{7}$$

From here it is easy to conclude that

$$\binom{\alpha}{k+1} = \frac{\alpha - k}{k+1} \binom{\alpha}{k} \quad \text{and} \quad \binom{\alpha+1}{k+1} = \frac{\alpha+1}{k+1} \binom{\alpha}{k} \tag{8}$$

Hence

$$\binom{\alpha}{k} + \binom{\alpha}{k+1} = \left(1 + \frac{\alpha - k}{k+1}\right) \binom{\alpha}{k} = \frac{\alpha+1}{k+1} \binom{\alpha}{k} = \binom{\alpha+1}{k+1} \tag{9}$$

for any real α and any integer $k \geq 0$,

If α is a positive integer, then equation (7) agrees with the well-known definition of the binomial coefficients. Even if α is not an integer, the definition is used in describing the binomial series

$$(1 + x)^{\alpha} = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k \qquad (|x| < 1).$$

In case α is a positive integer then $\binom{\alpha}{k} = 0$ for $k > \alpha$, since the factor $\alpha - j$ is zero in the numerator on the right-hand side of (7) for $j = \alpha$, and the above equation becomes the binomial formula. For positive integers α, equation (9) is the well-known property used in building the Pascal triangle, saying that each entry in the Pascal triangle is the sum of the two neighboring entries above it.

Given positive integers k and m with $m \geq k$ and a real α, using equation (9) we can see that

$$\begin{aligned} \sum_{n=k}^{m} \binom{\alpha+n}{k} &= \sum_{n=k}^{m} \left(\binom{\alpha+n+1}{k+1} - \binom{\alpha+n}{k+1} \right) \\ &= \binom{\alpha+m+1}{k+1} - \binom{\alpha+k}{k+1}. \end{aligned}$$

In case $\alpha = 0$, this becomes

$$\sum_{n=k}^{m} \binom{n}{k} = \binom{m+1}{k+1} - \binom{k}{k+1} = \binom{m+1}{k+1},$$

where the last equation holds since $\binom{k}{k+1} = 0$. This equation is easily shown to be equivalent to equation (4) after some change of notation.