TOTALLY BOUNDED SPACES[1]

Our goal here is to give a characterization of compact metric spaces. To this end we introduce the following concept.

Definition. *The metric space (E, d) is called* totally bounded *if for every $\epsilon > 0$ it can be covered by finitely many closed balls of radius ϵ.*

It will be convenient, but not essential, that we said closed balls above, since, clearly, if E can be covered by finitely many closed balls of radius $\epsilon/2$, then the open balls with the same center and radius ϵ will also cover E. One can also observe that if (E, d) and $S \subset E$, then the subspace (S, d) is also totally bounded. This is almost obvious, but there is the minor hitch: if we cover E by finitely many closed balls of radius ϵ, then we may not be able use these same closed balls to cover S, since the centers of these closed balls may not all be in S, and so they will not be closed balls in (S, d). Cover instead E by finitely many closed balls of radius $\epsilon/2$, then take only those balls B for which $B \cap S \neq \emptyset$, take $p \in B \cap S$, and replace B with a closed ball B' in S with center p and radius ϵ. As $B \cap S \subset B'$, it is clear that these new closed balls will cover S.

A space metric space is called *bounded* if it can be included in a single open ball. It is clear that a totally bounded space is also bounded, since finitely many closed balls can always be covered by a single open ball; the converse is, however, not true. For any integer $n \geq 1$ let $\mathbb{R}^n$ be the set $\{(x_1, \dots, x_n) : x_i \in \mathbb{R} \text{ for } 1 \leq i \leq n\}$; for $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{x} = (x_1, \dots, x_n)$ write $d(\mathbf{x}, \mathbf{y}) = \left(\sum_{i=1}^{n}(x_i - y_i)^2\right)^{1/2}$. Then one can show that $(\mathbb{R}^n, d)$ is a metric space (see Rosenlicht [1, pp. 34–36]); this space is called the *n-dimensional Euclidean space*. It is easy to show that any bounded subspace of the n-dimensional Euclidean space is totally bounded (see [1, p. 57]). Since this space is also complete (this is easy to conclude from the completeness of $\mathbb{R}$; cf. [1, p. 53]), it follows from the Theorem below that any closed and bounded subspace of the n-dimensional Euclidean space is compact.

The space l^2 is defined as the pair (E, d) where $E = \{(x_1, x_2, \dots) : x_i \in \mathbb{R} \text{ for } i \geq 1 \text{ and } \sum_{i=1}^{\infty} x_i^2 < +\infty\}$ and the distance function d is defined as $d(\mathbf{x}, \mathbf{y}) = \left(\sum_{i=1}^{\infty}(x_i - y_i)^2\right)^{1/2}$ for $\mathbf{x} = (x_1, x_2, \dots)$ and $\mathbf{y} = (y_1, y_2, \dots)$ in E. It can be shown that l^2 is a complete metric space, and that no closed ball of positive radius in l^2 is totally bounded.

We will need a few more concepts.

Definition. *The metric space is called sequentially compact if every sequence in it has a convergent subsequence.*

Definition. *Let (E, d) be a metric space and let $p \in E$ and $S \subset E$. We say that p is a* cluster point *of S if every open ball with center p in E contains infinitely many elements of S.*

Observe that we can equivalently say that p is a cluster point of S if every open ball with center p contains at least one element of S different from p. Indeed, if the open ball with center p were to contain only finitely many elements of S, all these points, except p, would be excluded from an open ball with center p and an appropriately smaller radius. The following result is simple.

Lemma. *Let (E, d) be a metric space, and let S and T be sets such that $T \subset S \subset E$, S is compact, and T is infinite. Then T has a cluster point in S.*

Proof. Assume, on the contrary, that no point $p \in S$ is a cluster point of T. For each $p \in S$ we can then take an open ball B_p with center p such that $B_p \cap T$ is finite. We have

$$S \subset \bigcup_{p \in S} B_p.$$

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That is, these open balls cover S. By the compactness of S, there are finitely many among these open balls that also cover S. In other words, there is a finite set $S' \subset S$ such that

$$S \subset \bigcup_{p \in S'} B_p.$$

We then have

$$T \subset S \cap T \subset \Big(\bigcup_{p \in S'} B_p \Big) \cap T \subset \bigcup_{p \in S'} (B_p \cap T).$$

The sets $B_p \cap T$ for $p \in S'$ are finite; the union of finitely many of these is finite. This is, however, a contradiction, since T was assumed to be infinite, completing a proof.

Theorem. *Let (E, d) be a metric space. The following are equivalent:*

(i) *(E, d) is compact;*
(ii) *(E, d) is sequentially compact;*
(iii) *(E, d) is complete and totally bounded.*

Proof. We will prove this result by showing the implications (i) $\to$ (ii), (ii) $\to$ (iii), and (iii) $\to$ (i).

Proof of (i) $\to$ (ii). Assume (E, d) is compact. Let $\{p_n\}_{n=1}^{\infty}$ be a sequence of points $p_n \in E$. If the set $\{p_n : n \in \mathbb{N}\}$ is finite, then there is a $p \in E$ such that $p = p_n$ for infinitely many n's. Then we can take a subsequence $\{p_{n_k}\}_{k=1}^{\infty}$ of $\{p_n\}_{n=1}^{\infty}$ such that $p_{n_k} = p$ for all $k \in \mathbb{N}$. This subsequence converges to p.

If the set $\{p_n : n \in \mathbb{N}\}$ is infinite, then it has a cluster point p by the above Lemma. For each $k \in \mathbb{N}$ the open ball $U(p, 1/k)$ contains infinitely elements of the set $\{p_n : n \in \mathbb{N}\}$. Let $n_k \in \mathbb{N}$ be such that $p_{n_k} \in U(p, 1/k)$ and, in case $k > 1$, we also have $n_k > n_{k-1}$. Then the subsequence $\{p_{n_k}\}_{k=1}^{\infty}$ converges to p.

Proof of (ii) $\to$ (iii). Assume (E, d) is sequentially compact. We will first show that (E, d) is complete. Let $\{p_n\}_{n=1}^{\infty}$ be a Cauchy sequence; then this sequence has a convergent subsequence. A Cauchy sequence that has a convergent subsequence is itself convergent; so $\{p_n\}_{n=1}^{\infty}$ itself is convergent, showing that (E, d) is complete.

Next we show that (E, d) is totally bounded. Assume, on the contrary that it is not totally bounded; let $\epsilon > 0$ be such that (E, d) cannot be covered by finitely many closed balls of radius ϵ. Select a sequence p_n of points $p_n \in E$ such that, writing $\bar{U}(p, \epsilon)$ for the closed ball of center p and radius ϵ, when selecting p_n we have

$$p_n \notin \bigcup_{k=1}^{n-1} \bar{U}(p_k, \epsilon);$$

since the closed balls on the right-hand side do not cover E according to our assumptions, it is possible to select p_n in such a way. Then $d(p_n, p_m) > \epsilon$ for every two distinct $n, m \in \mathbb{N}$, showing that the sequence $\{p_n\}_{n=1}^{\infty}$ is not Cauchy sequence, and so it does not converge. This contradicts the assumption that (E, d) is sequentially compact, showing that (E, d) is totally bounded.

(iii) $\to$ (i). Assume (E, d) is complete and totally bounded, and assume, on the contrary, that E is not compact. Let $\mathcal{U}$ be a collection of open sets such that $E \subset \bigcup \mathcal{U}$ (in this case we actually have $E = \bigcup \mathcal{U}$, since E is the whole space) and there is no finite $\mathcal{U}' \subset \mathcal{U}$ for which $E \subset \bigcup \mathcal{U}'$, i.e., that E cannot be covered by finitely many sets in $\mathcal{U}$. We will construct a sequence of closed balls B_n for $n \geq 0$ such that $B_0 = E$ (since E is totally bounded, it is also bounded, so E can be regarded as a closed ball), for $n \geq 0$ the closed ball B_n cannot be covered by finitely many elements of $\mathcal{U}$, i.e., for $n \geq 0$ there is no finite set $\mathcal{U}' \subset \mathcal{U}$ with $B_n \subset \bigcup \mathcal{U}'$, and such that, for each $n \geq 1$, the radius of B_n is $1/n$, and the center p_n of B_n is in B_{n-1}, i.e., $p_n \in B_{n-1}$,

Let $n \geq 1$ and assume that B_{n-1} has already been constructed in such a way that B_{n-1} cannot be covered by finitely many elements of $\mathcal{U}$ (this is certainly true if $n = 1$, since $B_0 = E$). To construct B_n for $n \geq 1$, cover B_{n-1} by finitely many closed balls of radius $1/n$ in such a way that the centers of each of these balls is in B_{n-1}; as we pointed out above, this is possible since (B_{n-1}, d), being the subspace of a totally bounded space, is totally bounded. There must be among these closed balls one that cannot be covered by finitely many elements of $\mathcal{U}$; select one of these balls as B_n. Let p_n be the center of B_n.

The sequence $\{p_n\}_{n=1}^{\infty}$ is a Cauchy sequence; indeed, for $m > n$ we have $p_m \in B_n = \bar{U}(p_n, 1/n)$, so $d(p_n, p_m) \leq 1/n$. Let p be the limit of this sequence. As B_n, being a closed ball, is a closed set for each

$n \in \mathbb{N}$ and $p_m \in B_n$ for all $m \geq n$, we have $p \in B_n$ for each $n \in \mathbb{N}$. That is, $d(p, p_n) \leq 1/n$ for all $n \in \mathbb{N}$. Now, $p \in U$ holds for some $U \in \mathcal{U}$, since $\mathcal{U}$ covers E. As U is open, there is an $\epsilon > 0$ such that $U(p, \epsilon) \subset U$. Then, for $n \geq 3/\epsilon$ we have $3/n \leq \epsilon$, so $U(p, 3/n) \subset U(p, \epsilon) \subset U$. Then we also have $B_n = \bar{U}(p_n, 1/n) \subset \bar{U}(p, 2/n) \subset U(p, 3/n) \subset U = \bigcup\{U\}$; the first inclusion here holds since for any $q \in B_n$ we have $d(p, q) \leq d(p, p_n) + d(p_n, q) \leq 1/n + 1/n = 2/n$. That is, B_n can be covered by a one-element subset of $\mathcal{U}$. This is a contradiction, since we assumed that B_n cannot be covered by finitely many elements of $\mathcal{U}$. This contradiction shows that the assumption that (E, d) is not compact was wrong, completing the proof.

Topological spaces are generalizations of metric spaces; in topological spaces, there are open and closed sets, but there is no distance function. There are topological spaces that are sequentially compact but not compact; so, maintaining the distinction between the notions of compact and sequentially compact is important.

Reference

1. Maxwell Rosenlicht, *Introduction to Analysis*, Dover Publications, Inc., New York, 1986.