Blackboard exam 1, Mathematics Mathematics 4701, Section TY3
Starts: 4:30 pm, Thurs, Mar 4; ends: 5:20 pm (late submission loses points).
Instructor: Attila Máté

1.*a*) Find the Lagrange interpolation polynomial $f(x)$ such that $f(1) = 11$, $f(3) = 5$, $f(4) = 3$.

Solution. Write $x_1 = 1$, $x_2 = 3$, $x_3 = 4$. We have

$$l_1(x) = \frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)} = \frac{(x-3)(x-4)}{(1-3)(1-4)} = \frac{1}{6}(x-3)(x-4),$$

$$l_2(x) = \frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)} = \frac{(x-1)(x-4)}{(3-1)(3-4)} = -\frac{1}{2}(x-1)(x-4),$$

$$l_3(x) = \frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)} = \frac{(x-1)(x-3)}{(4-1)(4-3)} = \frac{1}{3}(x-1)(x-3).$$

Thus, we have

$$P(x) = f(1)l_1(x) + f(3)l_2(x) + f(4)l_3(x) = 11 \cdot \frac{1}{6}(x-3)(x-4) + 5 \cdot \left(-\frac{1}{2}\right)(x-1)(x-4)$$
$$+ 3 \cdot \frac{1}{3}(x-1)(x-3) = \frac{11}{6}(x-3)(x-4) - \frac{5}{2}(x-1)(x-4) + (x-1)(x-3)$$

b) Estimate the error of Lagrange interpolation when interpolating $f(x) = \ln x$ at $x = 5$, using the interpolation points $x_1 = 2$, $x_2 = 3$, and $x_3 = 6$.

Solution. Noting that the third derivative of $\ln x$ equals $2/x^3$, with $f(x) = \ln x$ and with some ξ between 2 and 6, for the error at $x = 5$ we have

$$E(x) = f'''(\xi)\frac{(x-2)(x-3)(x-6)}{3!} = \frac{2}{\xi^3}\frac{(5-2)(5-3)(5-6)}{6} = -\frac{2}{\xi^3}$$

according to the error formula of the Lagrange interpolation, where ξ is some number in the interval spanned by x, x_1, x_2, and x_3, i.e., in the interval $(2, 6)$. Clearly, the right-hand side is smallest for $\xi = 2$ and largest for $x = 6$. Thus we have

$$-\frac{1}{4} < E(5) < -\frac{1}{108}.$$

We have strict inequalities, since the values $\xi = 2$ and $\xi = 6$ are not allowed.

2. Find the Newton-Hermite interpolation polynomial for $f(x)$ with $f(3) = 5$, $f'(3) = 6$, $f''(3) = -14$, $f(5) = 13$, $f'(5) = -2$.

a) First, write the divided difference table, using the points 3, 5 in natural order.

Solution. We have $f[x] = f(x)$; hence $f[3] = 4$ and $f[5] = 13$. Further, $f[x, x] = f'(x)$; hence $f[3, 3] = 6$ and $f[5, 5] = -2$. Finally, $f[x, x, x] = (1/2)f''(x)$; so $f[3, 3, 3] = -7$. Next

$$f[3,5] = \frac{f[5]-f[3]}{5-3} = \frac{13-5}{2} = 4,$$

$$f[3,3,5] = \frac{f[3,5]-f[3,3]}{5-3} = \frac{4-6}{2} = -1,$$

and

$$f[3,5,5] = \frac{f[5,5]-f[3,5]}{5-3} = \frac{-2-4}{2} = -3.$$

Therefore,

$$f[3,3,3,5] = \frac{f[3,3,5]-f[3,3,3]}{5-3} = \frac{-1-(-7)}{2} = 3,$$

$$f[3,3,5,5] = \frac{f[3,5,5] - f[3,3,5]}{5-3} = \frac{-3-(-1)}{2} = -1,$$

and

$$f[3,3,3,5,5] = \frac{f[3,3,5,5] - f[3,3,3,5]}{5-3} = \frac{-1-3}{2} = -2.$$

Summarizing this in a divided difference table, we have

$$\begin{array}{cccccc}
x & f[.] & f[.,.] & f[.,.,.] & f[x.,.,.,.] & f[.,.,.,.,.] \\
 & & & & & \\
3 & 5 & & & & \\
 & & 6 & & & \\
3 & 5 & & -7 & & \\
 & & 6 & & 3 & \\
3 & 5 & & -1 & & -2 \\
 & & 4 & & -1 & \\
5 & 13 & & -3 & & \\
 & & -2 & & & \\
5 & 13 & & & &
\end{array}$$

b) Using the divided difference table, write the Newton-Hermite interpolation polynomial using the order of points 3, 3, 3, 5, 5.

Solution. We have

$$\begin{aligned}
P(x) &= f[3] + f[3,3](x-3) + f[3,3,3](x-3)(x-3) + f[3,3,3,5](x-3)(x-3)(x-3) \\
&\quad + f[3,3,3,5,5](x-3)(x-3)(x-3)(x-5) \\
&= 5 + 6(x-3) - 7(x-3)^2 + 3(x-3)^3 - 2(x-3)^3(x-5).
\end{aligned}$$

c) Using the divided difference table, write the Newton-Hermite interpolation polynomial using the order of points 5, 3, 3, 5, 3.

Solution. We have

$$\begin{aligned}
P(x) &= f[5] + f[5,3](x-5) + f[5,3,3](x-5)(x-3) + f[5,3,3,5](x-5)(x-3)(x-3) \\
&\quad + f[5,3,3,5,3](x-5)(x-3)(x-3)(x-5) \\
&= 13 + 4(x-5) - (x-5)(x-3) - (x-5)(x-3)^2 - 2(x-5)^2(x-3)^2.
\end{aligned}$$