Blackboard exam 2, Mathematics Mathematics 4701, Section TY3 Starts: 4:30 pm, Thurs, Mar 18; ends: 5:20 pm (late submission loses points). Instructor: Attila Máté

1. Evaluate

$$(2x - y^3)^2$$

for  $x = 5 \pm 0.004$  and  $y = 2 \pm 0.007$ .

**Solution.** There is no problem with the actual calculation. With x = 5 and y = 2 we have

$$(2x - y^3)^2 = (10 - 8)^2 = 4.$$

The real question is, how accurate this result is? Writing

$$f(x,y) = (2x - y^3)^2,$$

we estimate the error of f by its total differential

$$df(x,y) = \frac{\partial f(x,y)}{\partial x}dx + \frac{\partial f(x,y)}{\partial y}dy = 4(2x - y^3) \, dx - 6y^2(2x - y^3) \, dy,$$

where x = 5, y = 2, and  $dx = \pm 0.004$  and  $dy = \pm 0.007$ , that is,  $|dx| \le 0.004$  and  $|dy| \le 0.007$ .<sup>1</sup> Thus

$$|df(x,y)| \le |4(2x-y^3)| |dx| + |-6y^2(2x-y^3)| |dy|$$
  
$$\lesssim 4 \cdot 2 \cdot .004 + 6 \cdot 4 \cdot 2 \cdot .007 = .032 + .336 = .368$$

Thus  $f(x,y) \approx 4 \pm .368$ . This example shows that the precision of the result can be much worse than might be expected on the basis of the precision of the input values.

2.a) Consider the equation f(x) = 0 with  $f(x) = x^3 - 7x + 3$ . Using Newton's method with  $x_0 = 2$  as a starting point, find the next approximation to the solution of the equation.

Solution. We have

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = x_0 - \frac{x_0^3 + 7x_0 + 3}{3x_0^2 - 7} = 2 - \frac{8 - 14 + 3}{12 - 7} = 2 + \frac{3}{5} = 2.6.$$

The actual solution is approximately 2.397661540892259. The approximation 2.6 is only slightly better than our initial approximation 2, but after this, the successive approximations will improve fast. In fact, the next few approximations are 2.421084337349398, 2.398036787813038, 2.397661639700024, 2.397661540892265, 2.397661540892259.

b) Evaluate the derivative of  $P(x) = 2x^3 - 3x^2 + 2x - 5$  at x = 2 using Horner's method. Show the details of your calculation.

**Solution.** We have  $a_0 = 2$ ,  $a_1 = -3$ ,  $a_2 = 2$ ,  $a_3 = -5$ , and  $x_0 = 2$ . Further, we have  $b_0 = a_0$  and  $b_k = a_k + b_{k-1}x_0$  for k with  $0 < k \le 3$ . Therefore,

$$b_0 = a_0 = 2,$$
  

$$b_1 = a_1 + b_0 x_0 = -3 + 2 \cdot 2 = 1,$$
  

$$b_2 = a_2 + b_1 x_0 = 2 + 1 \cdot 2 = 4,$$
  

$$b_3 = a_3 + b_2 x_0 = -5 + 4 \cdot 2 = 3.$$

<sup>&</sup>lt;sup>1</sup>It is more natural to write  $\Delta x$  and  $\Delta y$  for the errors of x and y, but in the total differential below one customarily uses dx and dy.

Actually, we did not need to calculate  $b_3$ , since it is not used in calculating the derivative. The derivative as x = 2 is the value for x = 2 of the polynomial  $b_0x^2 + b_1x + b_2$ . Using Horner's rule, this can be calculated by first calculating the coefficients  $c_0 = b_0$  and  $c_k = b_k + c_{k-1}x_0$  for k with  $0 < k \le 2$ , and then the value of the polynomial being considered will be  $c_2$ . That is,

$$c_0 = b_0 = 2,$$
  
 $c_1 = b_1 + c_0 x_0 = 1 + 2 \cdot 2 = 5,$   
 $c_2 = b_2 + c_1 x_0 = 4 + 5 \cdot 2 = 14.$ 

That is,  $P'(2) = c_2 = 14$ . It is easy to check that this result is correct. There is no real saving when the calculation is done for a polynomial of such low degree. For higher degree polynomials, there is definitely a saving in calculation. Another advantage of the method, especially for computers, is that the formal differentiation of polynomials can be avoided.

3. We want to evaluate

$$\int_{2}^{3} \frac{dx}{\ln x}$$

using the composite trapezoidal rule with five decimal precision, i.e., with an error not exceeding  $5 \cdot 10^{-6}$ . What value of *n* should one use when dividing the interval [2,3] into *n* parts?

**Solution.** The error term in the composite Simpson formula when integrating f on the interval [a, b] and dividing the interval into 2n parts is

$$-\frac{(b-a)^3}{12n^2}f''(\xi).$$

We want to use this with a = 0, b = 1, and  $f(x) = 1/\ln x$ . We have

$$f''(x) = \frac{1}{x^2} \left( \frac{1}{(\ln x)^2} + \frac{2}{(\ln x)^3} \right).$$

It is clear that f''(x) is decreasing on the interval  $(1, +\infty)$ , so it assumes its maximum of the interval [2,3] at x = 2. We have  $f''(2) \approx 2.02173$ . So, noting that a = 2 and b = 3, the absolute value of the error is

$$\frac{(b-a)^3}{12n^2}|f''(\xi)| = \frac{1}{12n^2}|f''(\xi)| \lesssim \frac{1}{12n^2} \cdot 2.02173 \lesssim \frac{0.168478}{n^2}.$$

In order to ensure that this error is less than  $5 \cdot 10^{-6}$ , we need to have  $0.168478/n^2 < 5 \cdot 10^{-6}$ , i.e.,

$$n > \sqrt{\frac{0.168478}{5} \cdot 10^6} \approx \sqrt{33695.5} = 183.563$$

So one needs to make sure that  $n \ge 183.563$ . Thus one needs to divide the interval [2, 3] into (at least) 184 parts in order to get the result with 4 decimal precision while using the trapezoidal rule.