Blackboard exam 3, Mathematics Mathematics 4701, Section TY3
Starts: 4:30 pm, Thurs, Apr 8; ends: 5:20 pm (late submission loses points).
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1. Given a certain function f, we are using the formula

$$\bar{f}(x,h) = \frac{f(x+h) - f(x-h)}{2h}$$

to approximate its derivative. We have

$$\bar{f}(2, 1/8) \approx 2.842,533 \quad \text{and} \quad \bar{f}(2, 1/16) \approx 2.871,272.$$

Using Richardson extrapolation, find a better approximation for $f'(2)$.

Solution. We have

$$f'(x) = \bar{f}(x,h) + c_1 h^2 + c_2 h^4 \dots$$
$$f'(x) = \bar{f}(x,2h) + c_1 (2h)^2 + c_2 (2h)^4 \dots$$

with some c_1, c_2, ... Multiplying the first equation by 4 and subtracting the second one, we obtain

$$3f'(x) = 4\bar{f}(x,2h) - f(x,h) + 12c_2 h^4 + \dots .$$

That is, with $x = 2$ and $h = 1/16$ we have

$$f'(x) \approx \frac{4\bar{f}(x,h) - \bar{f}(x,2h)}{3} \approx \frac{4 \cdot 2.871,272 - 2.842,533}{3} = 2.880,852$$

The function in the example is $f(x) = -\sin \frac{12}{x}$, and $f'(2) \approx 2.880,511$.

2. Describe how to deal with the singularity in the integral

$$\int_0^1 x^{-1/2} e^{-x^2}\,dx$$

if one wants to evaluate this integral using Simpson's rule.

Solution. One can subtract the singularity by taking an initial segment of the Taylor series at $x = 0$ of e^{-x^2}. We have

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!};$$

this Taylor series is convergent on the whole real line. Therefore, we have

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2} + O(x^6),$$

where the symbol $O(\cdot)$ is meant as $x \to 0$. Hence

$$x^{-1/2}\left(e^{-x^2} - 1 + x^2 - \frac{x^4}{2}\right) = O(x^{6-1/2}) = O(x^{11/2}).$$

The fourth derivative of this near $x = 0$ is $O(x^{3/2})$; that is, the fourth derivative tends to zero when $x \searrow 0$. So the fourth derivative is bounded on $(0,1)$, the interval of integration; therefore, Simpson's rule can be used to calculate the integral. To sum up, we have

$$\int_0^1 x^{-1/2} e^{-x^2}\,dx = \int_0^1 x^{-1/2}\left(e^{-x^2} - 1 + x^2 - \frac{x^4}{2}\right)dx + \int_0^1 \left(x^{-1/2} - x^{3/2} + \frac{x^{7/2}}{2}\right)dx.$$

The first integral on the right-hand side can be calculated by Simpson's rule (at $x = 0$ take the integrand to be 0), and the second integral can be evaluated directly, by calculating the integral explicitly.

The value of the integral turns out to be approximately $1.689,677,189,514$.

Second Solution. The problem can also be solved by making the substitution $t = x^2$. With this substitution, we have $dt = 2x\,dx$, and so the integral becomes

$$\int_0^1 x^{-1/2}e^{-x^2}\,dx = \int_0^1 t^{-1}e^{-t^4}\,2t\,dt = 2\int_0^1 e^{-t^4}\,dt,$$

and the integral on the right-hand side has no singularity.

In fact, this method is quite general, and can be used instead of the subtraction of singularity method. If the singularity at 0 is of the order $x^{-\alpha}$ for α with $0 < \alpha < 1$, then the substitution $t = x^{1/(1-\alpha)}$ removes the singularity (if $\alpha \le 0$ then there is no singularity, and if $\alpha \ge 1$ then the integral is divergent).

3. Write a third order Taylor approximation for the solution $y(x)$ at $x = 1+h$ of the differential equation $y' = x + y^2$ with initial condition $y(1) = 2$ (i.e., the error term in expressing $y(1+h)$ should be $O(h^4)$).

Solution. We have $y(1) = 2$, $y'(1) = x + y^2 = 5$; the right-hand side was obtained by substituting $x = 1$ and $y = 2$. Differentiating, then using the equation $y' = x + y^2$, and again substituting $x = 1$ and $y = 2$, we obtain

$$y''(x) = (x + y^2)' = 1 + 2yy' = 1 + 2y(x + y^2) = 1 + 2xy + 2y^3 = 21.$$

Differentiating, then using the equation $y' = x + y^2$, and again substituting $x = 1$ and $y = 2$, we obtain

$$y'''(x) = (1 + 2xy + 2y^3)' = 2y + 2xy' + 6y^2y' = 2y + 2(x + 3y^2)y' = 2y + 2(x + 3y^2)(x + y^2) = 134.$$

$$\begin{aligned} y(1+h) &= y(1) + y'(1)h + y''(1)\frac{h^2}{2} + y'''(1)\frac{h^3}{6} + O(h^4) = 2 + 5h + 21\frac{h^2}{2} + 134\frac{h^3}{6} + O(h^4) \\ &= 2 + 5h + \frac{21}{2}h^2 + \frac{67}{3}h^3 + O(h^4) \end{aligned}$$

for h near 0.