Blackboard exam 4, Mathematics Mathematics 4701, Section TY3
Starts: 4:30 pm, Thurs, Apr 22; ends: 5:20 pm (late submission loses points).
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1. Assume that

$$c_1 n^5 \cdot 8^n + c_2 n \cdot 4^n + c_3 n^3 \cdot 9^n = 0$$

for all $n > 0$. Show that then $c_1 = 0$.

Solution. Assume $c_1 \neq 0$. We will write a difference operator that annihilates all but the first term in the above expression, while it reduces the first term to $c \cdot 8^n$, where c is a nonzero constant.

The difference operator

$$(E-8)^5$$

will lower the degree of the polynomial in the first term to 0 (i.e., it will change the term into $c \cdot 8^n$ with a nonzero c), while it will not change the degrees of the other polynomials. The difference operator

$$(E-4)^2$$

will annihilate the second term, while it will not change the degrees of the polynomials in the other terms. Finally, the difference operator

$$(E-9)^4$$

will annihilate the third term, while it will not change the degrees of the polynomials in the other terms. Hence the product of these differential operators,

$$(E-9)^5(E-4)^2(E-5)^4$$

will change the first term into $c \cdot 8^n$ with a nonzero c, and it will annihilate the second and the third terms.

Thus we obtain the equation

$$c \cdot 8^n = 0$$

for all $n > 0$, which can only hold if $c = 0$. Since $c \neq 0$ if $c_1 \neq 0$, this shows that we must have $c_1 = 0$

A similar argument can be used to show that we also have $c_2 = c_3 = 0$. Thus the terms $n^5 \cdot 8^n$, $n \cdot 4^n$, and $n^3 \cdot 9^n$ are linearly independent.

2. Given

$$A = LU = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 4 & 2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 8 & 4 \\ 0 & 3 & 6 \\ 0 & 0 & 7 \end{pmatrix},$$

write A as $L'U'$ such that L' is a lower triangular matrix and U' is an upper triangular matrix such that the elements in the main diagonal of U' are all 1's.

Solution. Write D for the diagonal matrix whose diagonal elements are the same as those of U. That is,

$$D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 7 \end{pmatrix}.$$

We have $A = LU = (LD)(D^{-1}U)$. Note that the inverse of a diagonal matrix is a diagonal matrix formed by the reciprocal of the elements in the diagonal:

$$D^{-1} = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/7 \end{pmatrix}.$$

Now, we have

$$LD = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 4 & 2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 7 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 6 & 3 & 0 \\ 8 & 6 & 7 \end{pmatrix}.$$

Furthermore,

$$D^{-1}U = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/7 \end{pmatrix} \begin{pmatrix} 2 & 8 & 4 \\ 0 & 3 & 6 \\ 0 & 0 & 7 \end{pmatrix} = \begin{pmatrix} 1 & 4 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus,

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 6 & 3 & 0 \\ 8 & 6 & 7 \end{pmatrix} \begin{pmatrix} 1 & 4 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}.$$

3. Consider the differential equation $y' = f(x,y)$ with initial condition $y(x_0) = y_0$. Show that with $x_1 = x_0 + h$ $(h > 0)$, the solution at x_1 can be obtained within an error of $O(h^3)$ by the formula

$$y_1 = y_0 + \frac{h}{2} f(x_0, y_0) + \frac{h}{2} f\big(x_0 + h, y_0 + hf(x_0, y_0)\big)$$

Solution. Writing f, f_x, and f_y for f and its partial derivatives at (x_0, y_0), we have

$$f(x_0 + h, y_0 + hf) = f + hf_x + hf\,f_y + O(h^2).$$

Substituting this into the formula for y_1, we obtain

$$y_1 = y_0 + \frac{h}{2} f + \frac{h}{2}(f + hf_x + hf\,f_y) + O(h^3) = y_0 + hf + \frac{h^2}{2}(f_x + f f_y) + O(h^3).$$

This agrees with a Taylor approximation of y_1 within an error of $O(h^3)$, which we wanted to show.

In other words, the above formula describes a Runge–Kutta method of order 2. This is the classical Runge–Kutta method of order 2.