Blackboard exam 5, Mathematics Mathematics 4701, Section TY3 Starts: 4:30 pm, Thurs, May 6; ends: 5:20 pm (late submission loses points). Instructor: Attila Máté

1. Show that the matrix

$$A = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

is positive definite.

Solution. Since the matrix A is symmetric, it is sufficient to show that its principal minors are positive. The 1×1 principal minor is 2. The 2×2 principal minor is the determinant

$$\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 2 \cdot 2 - 1 \cdot 1 = 3$$

is also positive. The 3×3 principal matrix is the determinant

$$\begin{vmatrix} 2 & 1 & -1 \\ 1 & 2 & -1 \\ -1 & -1 & 2 \end{vmatrix} = \begin{vmatrix} 0 & -1 & 3 \\ 0 & 1 & 1 \\ -1 & -1 & 2 \end{vmatrix} = (-1) \cdot \begin{vmatrix} -1 & 3 \\ 1 & 1 \end{vmatrix} = (-1)((-1) \cdot 1 - 3 \cdot 1) = 4;$$

here the first equation is obtained by adding the twice the third row to the first row and adding the third row to the second row. The second equation is then obtained by expanding the determinant by the first column. This shows that all principal minors of the symmetric matrix A are positive; hence A is positive definite, according to the theorem on p. 160 (pdf p. 166) in Section 35 of the notes.

2. Consider the matrix

$$A = \begin{pmatrix} 8 & -1 & 4 \\ 2 & 6 & 10 \\ 14 & -6 & 4 \end{pmatrix}.$$

An eigenvalue of this matrix is 2 with eigenvector $\mathbf{x} = (1, 2, -1)^T$. Do one step of Wielandt deflation.

Solution. We will use

$$\mathbf{z} = \frac{1}{2 \cdot 2} (2, 6, 10)^T = \frac{1}{2} (1, 3, 5)^T;$$

here the row vector (2, 6, 10) is the second row of the matrix A. The second row is used since the second component of the eigenvector **x** has the largest absolute value. In $1/(2 \cdot 2)$, the first 2 is the eigenvalue, the second 2 is the element with the largest absolute value of the eigenvector **x**.

The purpose of choosing the component of the largest absolute value of the eigenvector is to make the roundoff error the smallest possible. The reason this results in the smallest roundoff error is that this is likely to make the entries of the subtracted matrix $\mathbf{x} \cdot \mathbf{z}^T$ the smallest possible (since we divide by x_r , the largest component of \mathbf{x} , in calculating \mathbf{z}). The larger the components of the matrix that we are subtracting from A, the more the original values of the entries of A will perturbed by roundoff errors. This is especially important if one performs repeated deflations, since the roundoff errors then accumulate. When doing exact calculations with integers, this point may be moot, since in this case there are no roundoff errors.

We have

$$B = A - 2 \cdot \mathbf{x} \cdot \mathbf{z}^{T} = \begin{pmatrix} 8 & -1 & 4 \\ 2 & 6 & 10 \\ 14 & -6 & 4 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \begin{pmatrix} 1 & 3 & 5 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 8 & -1 & 4 \\ 2 & 6 & 10 \\ 14 & -6 & 4 \end{pmatrix} - \begin{pmatrix} 1 & 3 & 5 \\ 2 & 6 & 10 \\ -1 & -3 & -5 \end{pmatrix} = \begin{pmatrix} 7 & -4 & -1 \\ 0 & 0 & 0 \\ 15 & -3 & 9 \end{pmatrix}$$

When looking for the eigenvalues of the matrix B (except for the eigenvalue 0, which is not an eigenvalue of the original matrix A), one can delete the second row and second column of the matrix B, and look for the eigenvalues of

$$B' = \begin{pmatrix} 7 & -1 \\ 15 & 9 \end{pmatrix}.$$