1. a) Calculate \( x - \sqrt{x^2 - 2} \) for \( x = 1,000,000 \) with 6 significant digit accuracy. Avoid the loss of significant digits.

**Solution.** We cannot use the expression given directly, since \( x \) and \( \sqrt{x^2 - 1} \) are too close, and their subtraction will result in a loss of precision. To avoid this, note that

\[
x - \sqrt{x^2 - 2} = \left( x - \sqrt{x^2 - 2} \right) \cdot \frac{x + \sqrt{x^2 - 2}}{x + \sqrt{x^2 - 2}} = \frac{2}{x + \sqrt{x^2 - 2}}.
\]

To do the numerical calculation, it is easiest to first write that \( x = y \cdot 10^6 \), where \( y = 1 \). Then

\[
\frac{2}{x + \sqrt{x^2 - 2}} = \frac{2}{y + \sqrt{y^2 - 2} \cdot 10^{-12}} \cdot 10^{-6} = \frac{2}{1 + \sqrt{1 - 2 \cdot 10^{-12}} \cdot 10^{-6}}.
\]

The idea would be to do the rest of the calculation on computer. On paper, one might go a little further. First, note the Taylor expansion

\[
\sqrt{1 + t} = 1 + \frac{t}{2} - \frac{t^2}{8} + \frac{t^3}{16} - \ldots.
\]

Using this with \( t = -2 \cdot 10^{-12} \), we obtain that\(^2\)

\[
\sqrt{1 - 2 \cdot 10^{-12}} \approx 1 - 10^{-12} - 10^{-24}/2.
\]

Thus, the right-hand side of (1) approximately equals

\[
\frac{2}{1 + (1 - 10^{-12} - 10^{-24}/2)} \cdot 10^{-6} \approx \frac{2}{2 - 10^{-12}} \cdot 10^{-6} = \frac{1}{1 - 10^{-12}/2} \cdot 10^{-6}.
\]

To estimate the right-hand side here, use the Taylor expansion

\[
\frac{1}{1 - t} = 1 + t + t^2 + \ldots.
\]

This with \( t = 10^{-12} \) shows that the right-hand side of (2) approximately equals

\[
(1 + 10^{-12}/2 + 10^{-24}/4) \cdot 10^{-6} \approx 1.000,000,000 \cdot 10^{-6}.
\]

**Note.** Instead of rationalizing the denominator, as we did above, one can base the whole calculation on a Taylor series approximation. Using the Taylor series expansion of \( \sqrt{1 + t} \) given above, we have (writing \( x = 10^6 \), as before), that

\[
x - \sqrt{x^2 - 2} = 10^6 - \sqrt{10^{12} - 2} = 10^6 \cdot (1 - \sqrt{1 - 2 \cdot 10^{-12}})
\]

\[
= 10^6 \cdot (1 - (1 - 10^{-12} - 10^{-24}/2) - \ldots) \approx 1.000,000,000 \cdot 10^{-6}.
\]

Usually, the approach involving rationalizing the denominator preferred since it is more general; for example, if one wants to calculate \( x - \sqrt{x^2 - 1} \) for \( x = 100 \), the rationalizing approach would still work without any

\(^{1}\) All computer processing for this manuscript was done under Fedora Linux. \LaTeX was used for typesetting.

\(^{2}\) If one wants to show rigorously that the third term in the next line correctly represents the approximate error in the calculation, one may replace \(-t^2/8\) in the preceding line with the Lagrange remainder term for the Taylor series. Writing \( f(t) = \sqrt{1 + t} \), the term \(-t^2/8\) represents \( f^{(n)}(0)t^2/2! \), while the Lagrange remainder term would be \( f^{(n)}(\xi)t^2/2! = -(1 + \xi)^{-3/2}t^2/8 \) for some \( \xi \) between \( t = -2 \cdot 10^{-12} \) and 0 – clearly, the difference between these terms is negligible. Much the same comment applies to our estimate of \( 1/(1 - t) \) below.
change, while in the Taylor series approach one would need to calculate more terms to get a satisfactory precision.

b) Find $1 - \cos 0.009$ with 10 decimal digit accuracy.

**Solution.** Calculating $1 - \cos 0.009$ directly would lead to an unnecessary and unacceptable loss of accuracy. It is much better to use the Taylor series of $\cos x$ with $x = 9 \cdot 10^{-3}$:

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \ldots.$$  

For $|x| \leq 1$ this is an alternating series, and so, when summing finitely many terms of the series, the error will be less than the first omitted term. With $x = 9 \cdot 10^{-3}$, we have

$$\frac{x^6}{6!} < x^6 < 0.01^6 = 10^{-12},$$

so this term can be safely omitted. Thus, with $x$ as above, we have

$$1 - \cos 0.009 = 1 - \cos x \approx \frac{x^2}{2} - \frac{x^4}{24} \approx .000,040,500,000 - .000,000,000,273 = .000,040,499,727.$$  

2. a) Evaluate

$$\sqrt{x+y^2}$$

for $x = 5 \pm 0.04$ and $y = 2 \pm 0.08$.

**Solution.** There is no problem with the actual calculation. With $x = 5$ and $y = 2$ we have

$$\sqrt{x+y^2} = \sqrt{9} = 3.$$  

The real question is, how accurate this result is? Writing

$$f(x, y) = \sqrt{x+y^2},$$

we estimate the error of $f$ by its total differential

$$df(x, y) = \frac{\partial f(x, y)}{\partial x} dx + \frac{\partial f(x, y)}{\partial y} dy = \frac{1}{2\sqrt{x+y^2}} dx + \frac{y}{\sqrt{x+y^2}} dy,$$

where $x = 5$, $y = 2$, and $dx = \pm 0.04$ and $dy = \pm 0.08$, that is, $|dx| \leq 0.04$ and $|dy| \leq 0.08$. Thus

$$|df(x, y)| \leq \frac{1}{2\sqrt{x+y^2}} |dx| + \frac{y}{\sqrt{x+y^2}} |dy| \approx \frac{1}{2} \cdot 0.04 + \frac{2}{3} \cdot 0.08 = \frac{0.02 + 0.16}{3} = 0.06.$$  

Thus $f(x, y) = 3 \pm 0.06$.

b) The leading term of the Newton interpolation polynomial $P$ to a function $f$ with the nodes $x_0, x_1, \ldots, x_n$ is

$$f[x_0, x_1, \ldots, x_n]x^n.$$  

\[\text{It is more natural to write } \Delta x \text{ and } \Delta y \text{ for the errors of } x \text{ and } y, \text{ but in the total differential below one customarily uses } dx \text{ and } dy.\]
Using this, show that

\[
\begin{align*}
  f[x_0, x_1, \ldots, x_n] &= \frac{f^{(n)}(\xi)}{n!}
\end{align*}
\]

for some \( \xi \) in the interval spanned by \( x_0, x_1, \ldots, x_n \). (All the nodes \( x_0, x_1, \ldots, x_n \) are assumed to be distinct.)

**Solution.** Taking the \( n \)th derivative of the polynomial \( P \), only the derivative of the leading term survives. That is,

\[
P^{(n)}(x) = n!f[x_0, x_1, \ldots, x_n].
\]

On the other hand, \( f(x) - P(x) \) has at least \( n + 1 \) zeros, \( x_0, x_1, \ldots, x_n \). Hence \( f^{(n)}(x) - P^{(n)}(x) \) has at least one zero in the interval spanned by \( x_0, x_1, \ldots, x_n \). Writing \( \xi \) for such a zero, we have

\[
0 = f^{(n)}(\xi) - P^{(n)}(\xi) = f^{(n)}(\xi) - n!f[x_0, x_1, \ldots, x_n].
\]

Omitting the middle member of these equations and solving the remaining equality, we obtain

\[
f[x_0, x_1, \ldots, x_n] = \frac{f^{(n)}(\xi)}{n!}.
\]

as we wanted to show.

3. a) Find the Lagrange interpolation polynomial \( P(x) \) such that \( P(1) = -3 \), \( P(3) = -1 \), \( P(4) = 3 \).

**Solution.** Write \( x_1 = 1, x_2 = 3, x_3 = 4 \). We have

\[
\begin{align*}
  l_1(x) &= \frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)} = \frac{(x-3)(x-4)}{(1-3)(1-4)} = \frac{1}{6}(x-3)(x-4), \\
  l_2(x) &= \frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)} = \frac{(x-1)(x-4)}{(3-1)(3-4)} = \frac{1}{2}(x-1)(x-4), \\
  l_3(x) &= \frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)} = \frac{(x-1)(x-3)}{(4-1)(4-3)} = \frac{1}{3}(x-1)(x-3).
\end{align*}
\]

Thus, we have

\[
P(x) = P(1)l_1(x) + P(3)l_2(x) + P(4)l_3(x) = -3 \cdot \frac{1}{6}(x-3)(x-4) + (-1) \cdot \left( -\frac{1}{2} \right)(x-1)(x-4)
\]

\[
+ 3 \cdot \frac{1}{3}(x-1)(x-3) = -\frac{1}{2}(x-3)(x-4) + \frac{1}{2}(x-1)(x-4) + (x-1)(x-3) = x^2 - 3x - 1.
\]

b) Estimate the error of Lagrange interpolation when interpolating \( f(x) = 1/x \) at \( x = 2 \) when using the interpolation points \( x_1 = 1, x_2 = 4, \) and \( x_3 = 5 \).

**Solution.** Noting that the third derivative of \( 1/x \) equals \(-6/x^4\), with \( f(x) = 1/x \) and with some \( \xi \) between 1 and 5, for the error at \( x = 2 \) we have

\[
E(x) = f'''(\xi) \frac{(x-1)(x-4)(x-5)}{3!} = -\frac{6}{\xi^4} \frac{(2-1)(2-4)(2-5)}{6} = -\frac{6}{\xi^4}
\]

according to the error formula of the Lagrange interpolation, where \( \xi \) is some number in the interval spanned by \( x, x_1, x_2, \) and \( x_3, \) i.e., in the interval (1, 5). Clearly, the right-hand side is smallest for \( \xi = 1 \) and largest for \( x = 5 \). Thus we have

\[
-6 < E(5) < -\frac{6}{625}.
\]

We have strict inequalities, since the values \( \xi = 1 \) and \( \xi = 5 \) are not allowed.
4. Find the Newton-Hermite interpolation polynomial for \( f(x) \) with \( f(2) = 4, f'(2) = 15, f(4) = 10, f'(4) = 39, f''(4) = 28 \).

   a) First, write the divided difference table, using the points 2, 4 in natural order.

   **Solution.** We have \( f[x] = f(x) \); hence \( f[2] = 4 \) and \( f[4] = 10 \). Further, \( f[x, x] = f'(x) \); hence \( f[2, 2] = 15 \) and \( f[4, 4] = 39 \). Finally, \( f[x, x, x] = (1/2)f''(x) \); so \( f[4, 4, 4] = 14 \). Next

   \[
   f[2, 4] = \frac{f[4] - f[2]}{4 - 2} = \frac{10 - 4}{2} = 3,
   \]

   \[
   f[2, 2, 4] = \frac{f[2, 4] - f[2, 2]}{4 - 2} = \frac{3 - 15}{2} = -6,
   \]

   and

   \[
   f[2, 4, 4] = \frac{f[4, 4] - f[2, 4]}{4 - 2} = \frac{39 - 3}{2} = 18.
   \]

   Therefore,

   \[
   f[2, 2, 4, 4] = \frac{f[2, 4, 4] - f[2, 2, 4]}{4 - 2} = \frac{18 + 6}{2} = 12,
   \]

   and

   \[
   f[2, 4, 4, 4] = \frac{f[4, 4, 4] - f[2, 4, 4]}{4 - 2} = \frac{14 - 18}{2} = -2,
   \]

   and

   \[
   f[2, 2, 4, 4, 4] = \frac{f[2, 4, 4, 4] - f[2, 2, 4, 4]}{4 - 2} = \frac{-2 - 12}{2} = -7.
   \]

   We can summarize these values in a divided difference table:

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f[.] )</th>
<th>( f[,] )</th>
<th>( f[,])</th>
<th>( f[x, , ,] )</th>
<th>( f[x, , , ,] )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>4</td>
<td>15</td>
<td>-6</td>
<td>12</td>
<td>-7</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>10</td>
<td>3</td>
<td>18</td>
<td>-2</td>
</tr>
<tr>
<td>4</td>
<td>10</td>
<td>39</td>
<td>14</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>10</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

   b) Using the divided difference table, write the Newton-Hermite interpolation polynomial using the order of points 2, 2, 4, 4, 4.

   **Solution.** We have

   \[
P(x) = f[2] + f[2, 2](x - 2) + f[2, 2, 4](x - 2)(x - 2) + f[2, 2, 4, 4](x - 2)(x - 2)(x - 2)(x - 2)
   \]

   \[
   + f[2, 2, 4, 4, 4](x - 2)(x - 2)(x - 2)(x - 2)(x - 2)
   = 4 + 15(x - 2) - 6(x - 2)^2 + 12(x - 2)^2(x - 4) - 7(x - 2)^2(x - 4)^2.
   \]

   c) Using the divided difference table, write the Newton-Hermite interpolation polynomial using the order of points 4, 2, 4, 2, 4.

   **Solution.** We have

   \[
P(x) = f[4] + f[4, 2](x - 4) + f[4, 2, 4](x - 4)(x - 4) + f[4, 2, 4, 4](x - 4)(x - 4)(x - 4)(x - 4)
   \]

   \[
   + f[4, 2, 4, 4, 4](x - 4)(x - 4)(x - 4)(x - 4)(x - 4)
   = 10 + 3(x - 4) + 18(x - 2)(x - 4) + 12(x - 2)(x - 4)^2 - 7(x - 2)^2(x - 4)^2.
   \]
5. a) Consider the equation \( f(x) = 0 \) with \( f(x) = 2 - x + \ln x \). Using Newton’s method with \( x_0 = 3 \) as a starting point, find the next approximation to the solution of the equation.

**Solution.** We have

\[
x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = x_0 - \frac{2 - x_0 + \ln x_0}{\frac{1}{x_0} - 1} = 3 - \frac{-1 + \ln 3}{-2/3} = \frac{3 + 3 \ln 3}{2} \approx 3.14792.
\]

The actual solution is approximately 3.14619.

b) Evaluate the derivative of \( P(x) = x^3 - 4x^2 + 6x + 4 \) at \( x = 2 \) using Horner’s method. Show the details of your calculation.

**Solution.** We have \( a_0 = 1, \ a_1 = -4, \ a_2 = 6, \ a_3 = 4 \), and \( x_0 = 2 \). Further, we have \( b_0 = a_0 \) and \( b_k = a_k + b_{k-1}x_0 \) for \( k \) with \( 0 < k \leq 3 \). Therefore,

\[
b_0 = a_0 = 1,
\]
\[
b_1 = a_1 + b_0 x_0 = -4 + 1 \cdot 2 = -2,
\]
\[
b_2 = a_2 + b_1 x_0 = 6 + (-2) \cdot 2 = 2,
\]
\[
b_3 = a_3 + b_2 x_0 = 4 + 2 \cdot 2 = 8.
\]

Actually, we did not need to calculate \( b_3 \), since it is not used in calculating the derivative. The derivative as \( x = 2 \) is the value for \( x = 2 \) of the polynomial \( b_0 x^2 + b_1 x + b_2 \). Using Horner’s rule, this can be calculated by first calculating the coefficients \( c_0 = b_0 \) and \( c_k = b_k + c_{k-1} \) for \( k \) with \( 0 < k \leq 2 \), and then value of the polynomial being considered will be \( c_2 \). That is,

\[
c_0 = b_0 = 1,
\]
\[
c_1 = b_1 + c_0 x_0 = -2 + 1 \cdot 2 = 0,
\]
\[
c_2 = b_2 + c_1 x_0 = 2 + 0 \cdot 2 = 2.
\]

That is, \( P'(2) = c_2 = 2 \). It is easy to check that this result is correct. There is no real saving when the calculation is done for a polynomial of such low degree. For higher degree polynomials, there is definitely a saving in calculation. Another advantage of the method, especially for computers, is that the formal differentiation of polynomials can be avoided.

c) Let \( P \) and \( Q \) be polynomials, let \( x_0 \) and \( r \) be a numbers, and assume that

\[
P(x) = (x - x_0)Q(x) + r.
\]

Show that \( P'(x_0) = Q(x_0) \).

**Solution.** We have

\[
P'(x) = Q(x) + (x - x_0)Q'(x)
\]

simply be using the product rule for differentiation. Substituting \( x = x_0 \), we obtain that \( P'(x_0) = Q(x_0) \).

**Note:** The coefficients of the polynomial \( Q(x) \) can be produced by Horner’s method. By another use of Horner’s method, we can evaluate \( Q(x_0) \). This provides an efficient way to evaluate \( P'(x) \) on computers without using symbolic differentiation.