Exam 1, Mathematics 4701, Section TY2 21 copies
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1.*a*) Calculate $x-\sqrt{x^2-2}$ for $x=1{,}000{,}000$ with 6 significant digit accuracy. Avoid the loss of significant digits.

Solution. We cannot use the expression given directly, since x and $\sqrt{x^2-1}$ are too close, and their subtraction will result in a loss of precision. To avoid this, note that

$$x-\sqrt{x^2-2}=\left(x-\sqrt{x^2-2}\right)\cdot\frac{x+\sqrt{x^2-2}}{x+\sqrt{x^2-2}}=\frac{2}{x+\sqrt{x^2-2}}.$$

To do the numerical calculation, it is easiest to first write that $x=y\cdot 10^6$, where $y=1$. Then

$$\frac{2}{x+\sqrt{x^2-2}}=\frac{2}{y+\sqrt{y^2-2\cdot 10^{-12}}}\cdot 10^{-6}=\frac{2}{1+\sqrt{1-2\cdot 10^{-12}}}\cdot 10^{-6}. \tag{1}$$

The idea would be to do the rest of the calculation on computer. On paper, one might go a little further. First, note the Taylor expansion

$$\sqrt{1+t}=1+\frac{t}{2}-\frac{t^2}{8}+\frac{t^3}{16}-\dots.$$

Using this with $t=-2\cdot 10^{-12}$, we obtain that[2]

$$\sqrt{1-2\cdot 10^{-12}}\approx 1-10^{-12}-10^{-24}/2.$$

Thus, the right-hand side of (1) approximately equals

$$\frac{2}{1+(1-10^{-12}-10^{-24}/2)}\cdot 10^{-6}\approx\frac{2}{2-10^{-12}}\cdot 10^{-6}.=\frac{1}{1-10^{-12}/2}\cdot 10^{-6}. \tag{2}$$

To estimate the right-hand side here, use the Taylor expansion

$$\frac{1}{1-t}=1+t+t^2+\dots.$$

This with $t=10^{-12}$ shows that the right-hand side of (2) approximately equals

$$(1+10^{-12}/2+10^{-24}/4)\cdot 10^{-6}\approx 1.000{,}000{,}000\cdot 10^{-6}.$$

Note. Instead of rationalizing the denominator, as we did above, one can base the whole calculation on a Taylor series approximation. Using the Taylor series expansion of $\sqrt{1+t}$ given above, we have (writing $x=10^6$, as before), that

$$\begin{aligned}x-\sqrt{x^2-2}&=10^6-\sqrt{10^{12}-2}=10^6\cdot(1-\sqrt{1-2\cdot 10^{-12}})\\&=10^6\cdot(1-(1-10^{-12}-\frac{10^{-24}}{2}-\frac{10^{-36}}{2}-\dots)\approx 1.000{,}000{,}000\cdot 10^{-6}.\end{aligned}$$

Usually, the approach involving rationalizing the denominator preferred since it is more general; for example, if one wants to calculate $x-\sqrt{x^2-1}$ for $x=100$, the rationalizing approach would still work without any

[1] All computer processing for this manuscript was done under Fedora Linux. $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX was used for typesetting.

[2] If one wants to show rigorously that the third term in the next line correctly represents the approximate error in the calculation, one may replace $-t^2/8$ in the preceding line with the Lagrange remainder term for the Taylor series. Writing $f(t)=\sqrt{1+t}$, the term $-t^2/8$ represents $f''(0)t^2/2!$, while the Lagrange remainder term would be $f''(\xi)t^2/2!=-(1+\xi)^{-3/2}t^2/8$ for some ξ between $t=-2\cdot 10^{-12}$ and 0 – clearly, the difference between these terms is negligible. Much the same comment applies to our estimate of $1/(1-t)$ below.

change, while in the Taylor series approach one would need to calculate more terms to get a satisfactory precision.

b) Find $1 - \cos 0.009$ with 10 decimal digit accuracy.

Solution. Calculating $1 - \cos 0.009$ directly would lead to an unnecessary and unacceptable loss of accuracy. It is much better to use the Taylor series of $\cos x$ with $x = 9 \cdot 10^{-3}$:

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \ldots .$$

For $|x| \le 1$ this is an alternating series, and so, when summing finitely many terms of the series, the error will be less than the first omitted term. With $x = 9 \cdot 10^{-3}$, we have

$$\frac{x^6}{6!} < x^6 < 0.01^6 = 10^{-12},$$

so this term can be safely omitted. Thus, with x as above, we have

$$1 - \cos 0.009 = 1 - \cos x \approx \frac{x^2}{2} - \frac{x^4}{24} \approx .000,040,500,000 - .000,000,000,273 = .000,040,499,727.$$

2. *a*) Evaluate

$$\sqrt{x + y^2}$$

for $x = 5 \pm 0.04$ and $y = 2 \pm 0.08$.

Solution. There is no problem with the actual calculation. With $x = 5$ and $y = 2$ we have

$$\sqrt{x + y^2} = \sqrt{9} = 3.$$

The real question is, how accurate this result is? Writing

$$f(x, y) = \sqrt{x + y^2},$$

we estimate the error of f by its total differential

$$df(x, y) = \frac{\partial f(x, y)}{\partial x} dx + \frac{\partial f(x, y)}{\partial y} dy = \frac{1}{2\sqrt{x + y^2}}\, dx + \frac{y}{\sqrt{x + y^2}}\, dy,$$

where $x = 5$, $y = 2$, and $dx = \pm 0.04$ and $dy = \pm 0.08$, that is, $|dx| \le 0.04$ and $|dy| \le 0.08$.[3] Thus

$$\begin{aligned} |df(x, y)| &\le \left| \frac{1}{2\sqrt{x + y^2}} \right| |dx| + \left| \frac{y}{\sqrt{x + y^2}} \right| |dy| \\ &\lessapprox \frac{1}{2 \cdot 3} \cdot 0.04 + \frac{2}{3} \cdot 0.08 = \frac{0.02 + 0.16}{3} = 0.06. \end{aligned}$$

Thus $f(x.y) \approx 3 \pm 0.06$.

b) The leading term of the Newton interpolation polynomial P to a function f with the nodes x_0, x_1, $\ldots$ x_n is

$$f[x_0, x_1, \cdots, x_n]x^n.$$

[3] It is more natural to write Δx and Δy for the errors of x and y, but in the total differential below one customarily uses dx, and dy.

Using this, show that

$$f[x_0, x_1, \cdots, x_n] = \frac{f^{(n)}(\xi)}{n!}$$

for some ξ in the interval spanned by x_0, x_1, $\ldots$ x_n. (All the nodes x_0, x_1, $\ldots$ x_n are assumed to be distinct.)

Solution. Taking the nth derivative of the polynomial P, only the derivative of the leading term survives. That is,

$$P^{(n)}(x) = n! f[x_0, x_1, \cdots, x_n].$$

On the other hand, $f(x) - P(x)$ has at least $n+1$ zeros, x_0, x_1, $\ldots$ x_n. Hence $f^{(n)}(x) - P^{(n)}(x)$ has at least one zero in the interval spanned by x_0, x_1, $\ldots$ x_n. Writing ξ for such a zero, we have

$$0 = f^{(n)}(\xi) - P^{(n)}(\xi) = f^{(n)}(\xi) - n! f[x_0, x_1, \cdots, x_n].$$

Omitting the middle member of these equations and solving the remaining equality, we obtain

$$f[x_0, x_1, \cdots, x_n] = \frac{f^{(n)}(\xi)}{n!}.$$

as we wanted to show.

3. *a*) Find the Lagrange interpolation polynomial $P(x)$ such that $P(1) = -3$, $P(3) = -1$, $P(4) = 3$.

Solution. Write $x_1 = 1$, $x_2 = 3$, $x_3 = 4$. We have

$$l_1(x) = \frac{(x - x_2)(x - x_3)}{(x_1 - x_2)(x_1 - x_3)} = \frac{(x-3)(x-4)}{(1-3)(1-4)} = \frac{1}{6}(x-3)(x-4),$$

$$l_2(x) = \frac{(x - x_1)(x - x_3)}{(x_2 - x_1)(x_2 - x_3)} = \frac{(x-1)(x-4)}{(3-1)(3-4)} = -\frac{1}{2}(x-1)(x-4),$$

$$l_3(x) = \frac{(x - x_1)(x - x_2)}{(x_3 - x_1)(x_3 - x_2)} = \frac{(x-1)(x-3)}{(4-1)(4-3)} = \frac{1}{3}(x-1)(x-3).$$

Thus, we have

$$\begin{aligned} P(x) &= P(1)l_1(x) + P(3)l_2(x) + P(4)l_3(x) = -3 \cdot \frac{1}{6}(x-3)(x-4) + (-1) \cdot \left(-\frac{1}{2}\right)(x-1)(x-4) \\ &\quad + 3 \cdot \frac{1}{3}(x-1)(x-3) = -\frac{1}{2}(x-3)(x-4) + \frac{1}{2}(x-1)(x-4) + (x-1)(x-3) = x^2 - 3x - 1. \end{aligned}$$

b) Estimate the error of Lagrange interpolation when interpolating $f(x) = 1/x$ at $x = 2$ when using the interpolation points $x_1 = 1$, $x_2 = 4$, and $x_3 = 5$.

Solution. Noting that the third derivative of $1/x$ equals $-6/x^4$, with $f(x) = 1/x$ and with some ξ between 1 and 5, for the error at $x = 2$ we have

$$E(x) = f'''(\xi)\frac{(x-1)(x-4)(x-5)}{3!} = -\frac{6}{\xi^4}\frac{(2-1)(2-4)(2-5)}{6} = -\frac{6}{\xi^4}$$

according to the error formula of the Lagrange interpolation, where ξ is some number in the interval spanned by x, x_1, x_2, and x_3, i.e., in the interval $(1, 5)$. Clearly, the right-hand side is smallest for $\xi = 1$ and largest for $x = 5$. Thus we have

$$-6 < E(5) < -\frac{6}{625}.$$

We have strict inequalities, since the values $\xi = 1$ and $\xi = 5$ are not allowed.

4. Find the Newton-Hermite interpolation polynomial for $f(x)$ with $f(2) = 4$, $f'(2) = 15$, $f(4) = 10$, $f'(4) = 39$, $f''(4) = 28$.

a) First, write the divided difference table, using the points 2, 4 in natural order.

Solution. We have $f[x] = f(x)$; hence $f[2] = 4$ and $f[4] = 10$. Further, $f[x, x] = f'(x)$; hence $f[2, 2] = 15$ and $f[4, 4] = 39$. Finally, $f[x, x, x] = (1/2)f''(x)$; so $f[4, 4, 4] = 14$. Next

$$f[2,4] = \frac{f[4] - f[2]}{4-2} = \frac{10-4}{2} = 3,$$

$$f[2,2,4] = \frac{f[2,4] - f[2,2]}{4-2} = \frac{3-15}{2} = -6,$$

and

$$f[2,4,4] = \frac{f[4,4] - f[2,4]}{4-2} = \frac{39-3}{2} = 18.$$

Therefore,

$$f[2,2,4,4] = \frac{f[2,4,4] - f[2,2,4]}{4-2} = \frac{18+6}{2} = 12,$$

$$f[2,4,4,4] = \frac{f[4,4,4] - f[2,4,4]}{4-2} = \frac{14-18}{2} = -2,$$

and

$$f[2,2,4,4,4] = \frac{f[2,4,4,4] - f[2,2,4,4]}{4-2} = \frac{-2-12}{2} = -7.$$

We can summarize these values in a divided difference table:

x	$f[.]$	$f[.,.]$	$f[.,.,.]$	$f[x.,.,.,.]$	$f[x.,.,.,.,.]$
2	4				
		15			
2	4		-6		
		3		12	
4	10		18		-7
		39		-2	
4	10		14		
		39			
4	10				

b) Using the divided difference table, write the Newton-Hermite interpolation polynomial using the order of points 2, 2, 4, 4, 4.

Solution. We have

$$\begin{aligned} P(x) &= f[2] + f[2,2](x-2) + f[2,2,4](x-2)(x-2) + f[2,2,4,4](x-2)(x-2)(x-4) \\ &\quad + f[2,2,4,4,4](x-2)(x-2)(x-4)(x-4) \\ &= 4 + 15(x-2) - 6(x-2)^2 + 12(x-2)^2(x-4) - 7(x-2)^2(x-4)^2. \end{aligned}$$

c) Using the divided difference table, write the Newton-Hermite interpolation polynomial using the order of points 4, 2, 4, 2, 4.

Solution. We have

$$\begin{aligned} P(x) &= f[4] + f[4,2](x-4) + f[4,2,4](x-4)(x-2) + f[4,2,4,2](x-4)(x-2)(x-4) \\ &\quad + f[4,2,4,2,4](x-4)(x-2)(x-4)(x-2) \\ &= 10 + 3(x-4) + 18(x-2)(x-4) + 12(x-2)(x-4)^2 - 7(x-2)^2(x-4)^2. \end{aligned}$$

5. *a*) Consider the equation $f(x) = 0$ with $f(x) = 2 - x + \ln x$. Using Newton's method with $x_0 = 3$ as a starting point, find the next approximation to the solution of the equation.

Solution. We have

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = x_0 - \frac{2 - x_0 + \ln x_0}{\frac{1}{x_0} - 1} = 3 - \frac{-1 + \ln 3}{-2/3} = \frac{3 + 3\ln 3}{2} \approx 3.14792.$$

The actual solution is approximately 3.14619.

b) Evaluate the derivative of $P(x) = x^3 - 4x^2 + 6x + 4$ at $x = 2$ using Horner's method. Show the details of your calculation.

Solution. We have $a_0 = 1$, $a_1 = -4$, $a_2 = 6$, $a_3 = 4$, and $x_0 = 2$. Further, we have $b_0 = a_0$ and $b_k = a_k + b_{k-1}x_0$ for k with $0 < k \le 3$. Therefore,

$$b_0 = a_0 = 1,$$

$$b_1 = a_1 + b_0x_0 = -4 + 1 \cdot 2 = -2,$$

$$b_2 = a_2 + b_1x_0 = 6 + (-2) \cdot 2 = 2,$$

$$b_3 = a_3 + b_2x_0 = 4 + 2 \cdot 2 = 8.$$

Actually, we did not need to calculate b_3, since it is not used in calculating the derivative. The derivative as $x = 2$ is the value for $x = 2$ of the polynomial $b_0x^2 + b_1x + b_2$. Using Horner's rule, this can be calculated by first calculating the coefficients $c_0 = b_0$ and $c_k = b_k + c_{k-1}$ for k with $0 < k \le 2$, and then value of the polynomial being considered will be c_2. That is,

$$c_0 = b_0 = 1,$$

$$c_1 = b_1 + c_0x_0 = -2 + 1 \cdot 2 = 0,$$

$$c_2 = b_2 + c_1x_0 = 2 + 0 \cdot 2 = 2.$$

That is, $P'(2) = c_2 = 2$. It is easy to check that this result is correct. There is no real saving when the calculation is done for a polynomial of such low degree. For higher degree polynomials, there is definitely a saving in calculation. Another advantage of the method, especially for computers, is that the formal differentiation of polynomials can be avoided.

c) Let P and Q be polynomials, let x_0 and r be a numbers, and assume that

$$P(x) = (x - x_0)Q(x) + r.$$

Show that $P'(x_0) = Q(x_0)$.

Solution. We have

$$P'(x) = Q(x) + (x - x_0)Q'(x)$$

simply be using the product rule for differentiation. Substituting $x = x_0$, we obtain that $P'(x_0) = Q(x_0)$.

Note: The coefficients of the polynomial $Q(x)$ can be produced by Horner's method. By another use of Horner's method, we can evaluate $Q(x_0)$. This provides an efficient way to evaluate $P'(x)$ on computers without using symbolic differentiation.