Follow these instructions carefully:
Work on the paper provided; do not use your own paper. Work only on one problem on each sheet (you should not work on two different problems on the two sides of the same sheet). On the top of each page, print your name (encircle your last name) and indicate the number of the problem you are working on by writing e.g. “Problem #4”. Always encircle your final answer. If there are several parts to a problem, always indicate the part that you are answering, e.g. by writing “Answer to Part b)” (the number of the problem should be on the top of the page). Do not use a red pen or a red pencil. Do not write in the corner covered up by the staple (top left corner on the front side, top right corner on the back side). Each problem is worth the same amount of credit. Show all your work.

1. a) Consider the formula
\[ f(x, h) = \frac{f(x + h) - f(x - h)}{2h} \]
to approximate the derivative of a function \( f \). Assume we are able to evaluate \( f \) with about 5 decimal precision. Assume, further, that \( f'''(1) \approx 1 \). What is the best value of \( h \) to approximate the derivative?

b) Given a certain function \( f \), we are using the formula
\[ f(x, h) = \frac{f(x + h) - f(x - h)}{2h} \]
to approximate its derivative. We have
\[ f(1, 0.1) = 5.135, 466, 136 \quad \text{and} \quad f(1, 0.2) = 5.657, 177, 752 \]
Using Richardson extrapolation, find a better approximation for \( f'(1) \).

2. a) The equation \( e^x - x^2 - 4 = 0 \) has one solution; a good approximation to this solution is 2.16. Find a way to improve this approximation by fixed-point iteration.

b) State the usual sufficient condition for the fixed-point iteration to converge when solving the equation \( x = f(x) \).

3. We want to evaluate
\[ \int_0^1 e^{x^2} \, dx \]
using the composite trapezoidal rule with three decimal precision, i.e., with an error not exceeding \( 5 \cdot 10^{-4} \). What value of \( n \) should one use when dividing the interval \([0, 1]\) into \( n \) parts?

4. a) Let \( h \) and \( k \) be numbers, and let \( f(x, y) \) be a function that is differentiable sufficiently many times (so all required derivatives exist, and the order of mixed derivatives is interchangeable). Evaluate
\[ \left( h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^3 f. \]
In your answer, you may write \( f_x, f_{xy}, f_{yy}, \ldots \), for the various derivatives of \( f \).

b) Let \( f(x, y) \) be a function that is differentiable sufficiently many times. Evaluate
\[ \left( \frac{\partial}{\partial x} + f \frac{\partial}{\partial y} \right)^2 f. \]

5. Consider the differential equation \( y' = f(x, y) \) with initial condition \( y(x_0) = y_0 \). Show that, with \( x_1 = x_0 + h \), the solution at \( x_1 \) can be obtained with an error \( O(h^3) \) by the formula
\[ y_1 = y_0 + \frac{h}{4} f(x_0, y_0) + \frac{3h}{4} f \left( x_0 + \frac{2h}{3}, y_0 + \frac{2h}{3} f(x_0, y_0) \right) \]
In other words, this formula describes a Runge-Kutta method of order 2.