Exam 2, Mathematics 4701, Section TY2 — 21 copies
1:25 pm–3:25 pm, March 28, 2019, IH-137
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Follow these instructions carefully:
Work on the paper provided; do not use your own paper. *Work only on one problem on each sheet (you should not work on two different problems on the two sides of the same sheet).* On the top of each page, *print* your name (*encircle your last name*) and indicate the number of the problem you are working on by writing e.g. *"Problem #4"*. Always *encircle* your final answer. If there are several parts to a problem, always indicate the part that you are answering, e.g. by writing *"Answer to Part b)"* (the number of the problem should be on the top of the page). Do not use a *red* pen or a *red* pencil. Do not write in the corner covered up by the staple (top left corner on the front side, top right corner on the back side). Each problem is worth the *same* amount of credit. **Show all your work.**

1. *a*) Consider the formula

$$\bar{f}(x,h)=\frac{f(x+h)-f(x-h)}{2h}$$

to approximate the derivative of a function f. Assume we are able to evaluate f with about 5 decimal precision. Assume, further, that $f'''(1)\approx 1$. What is the best value of h to approximate the derivative?

b) Given a certain function f, we are using the formula

$$\bar{f}(x,h)=\frac{f(x+h)-f(x-h)}{2h}$$

to approximate its derivative. We have

$$\bar{f}(1,0.1)=5.135,466,136 \quad\text{and}\quad \bar{f}(1,0.2)=5.657,177,752$$

Using Richardson extrapolation, find a better approximation for $f'(1)$.

2. *a*) The equation $e^x-x^2-4=0$ has one solution; a good approximation to this solution is 2.16. Find a way to improve this approximation by fixed-point iteration.

b) State the usual sufficient condition for the fixed-point iteration to converge when solving the equation $x=f(x)$.

3. We want to evaluate

$$\int_0^1 e^{x^2}\,dx$$

using the composite trapezoidal rule with three decimal precision, i.e., with an error not exceeding $5\cdot 10^{-4}$. What value of n should one use when dividing the interval $[0,1]$ into n parts?

4. *a*) Let h and k be numbers, and let $f(x,y)$ be a function that is differentiable sufficiently many times (so all required derivatives exist, and the order of mixed derivatives is interchangeable). Evaluate

$$\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^3 f.$$

In your answer, you may write f_x, f_{xy}, f_{yy}, ..., for the various derivatives of f.

b) Let $f(x,y)$ be a function that is differentiable sufficiently many times. Evaluate

$$\left(\frac{\partial}{\partial x}+f\frac{\partial}{\partial y}\right)^2 f.$$

5. Consider the differential equation $y'=f(x,y)$ with initial condition $y(x_0)=y_0$. Show that, with $x_1=x_0+h$, the solution at x_1 can be obtained with an error $O(h^3)$ by the formula

$$y_1=y_0+\frac{h}{4}f(x_0,y_0)+\frac{3h}{4}f\left(x_0+\frac{2h}{3},y_0+\frac{2h}{3}f(x_0,y_0)\right).$$

In other words, this formula describes a Runge-Kutta method of order 2.