21 copies

Exam 3, Mathematics 4701, Section TY2 1:25 pm-3:25 pm, May 2, 2019, IH-137 Instructor: Attila Máté<sup>1</sup>

1.a) Given that  $y_0 = 1$ ,  $y_1 = 8$  and  $y_{n+2} = 4y_{n+1} - 4y_n$  for every integer  $n \ge 0$ , write a formula expressing  $y_n$ .

**Solution.** The characteristic equation of the recurrence equation  $y_{n+2} = 4y_{n+1} - 4y_n$  is  $\zeta^2 = 4\zeta - 4$ , i.e.,  $\zeta^2 - 4\zeta + 4 = 0$ , that is  $(\zeta - 2)^2 = 0$ . The only solution of this equation is  $\zeta = 2$  with multiplicity 2. Thus, the general solution of the above recurrence equation is

$$y_n = C_1 2^n + C_2 n 2^n.$$

The initial conditions  $y_0 = 1$  and  $y_1 = 8$  lead to the equations

$$C_1 = 1$$

and

$$2C_1 + 2C_2 = 8.$$

Substituting the first equation into the second equation, we can see that  $C_2 = 3$ . Thus the solution of the given recurrence equation is  $y_n = 2^n + 3n2^n$ , that is

$$y_n = (1+3n)2^n.$$

b) Write a difference operator that annihilates all but the first term in the expression

$$c_1 n^7 \cdot 9^n + c_2 n^3 \cdot 6^n + c_3 n^5 \cdot 7^n,$$

while it reduces the first term to  $c \cdot 9^n$ , where c is a nonzero constant (it is assumed that  $c_1 \neq 0$ ).

Solution. The difference operator

 $(E-9)^{7}$ 

will lower the degree of the polynomial in the first term to 0 (i.e., it will change the term into  $c \cdot 3^n$  with a nonzero c), while it will not change the degrees of the other polynomials. The difference operator

$$(E-6)^4$$

will annihilate the second term, while it will not change the degrees of the polynomials in the other terms. Finally, the difference operator

$$(E-7)^{6}$$

will annihilate the third term, while it will not change the degrees of the polynomials in the other terms. Hence the product of these differential operators,

$$(E-9)^7(E-6)^4(E-7)^6$$

will change the first term into  $c \cdot 3^n$  with a nonzero c, and it will annihilate the second and the third terms.

This argument can be used to show that if

$$c_1 n^7 \cdot 9^n + c_2 n^3 \cdot 6^n + c_3 n^5 \cdot 7^n,$$

then we must have  $c_1 = 0$ . Similar arguments can be used to show that we must also have  $c_2 = 0$  and  $c_3 = 0$ ; hence the terms  $n^7 \cdot 9^n$ ,  $n^3 \cdot 6^n$ , and  $n^5 \cdot 7^n$  are linearly independent.

<sup>&</sup>lt;sup>1</sup>All computer processing for this manuscript was done under Debian Linux.  $\mathcal{A}_{MS}$ -T<sub>E</sub>X was used for typesetting. The programming language Perl and the computer algebra system Maxima was used in creating some of the problems.

2. Solve the equation

$$\begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 2 & 2 & 1 \end{pmatrix} \begin{pmatrix} 5 & 4 & 3 \\ 0 & 4 & 1 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5 \\ 14 \\ -6 \end{pmatrix}.$$

Solution. Write the above equation as  $LU\mathbf{x} = \mathbf{b}$ , and write  $\mathbf{y} = U\mathbf{x}$ . Then this equation can be written as  $L\mathbf{y} = \mathbf{b}$ , that is

$$\begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 2 & 2 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 5 \\ 14 \\ -6 \end{pmatrix}.$$

This equation can be written as

$$y_1 = 5 
4y_1 + y_2 = 14 
2y_1 + 2y_2 + y_3 = -6$$

The solution of these equations is straightforward (the method for solving them is called forward substitution). From the first equation, we have  $y_1 = 5$ . Then, from the second equation we have  $y_2 = 14 - 4y_1 = 14 - 4 \cdot 5 = -6$ . Finally, from the third equation  $y_3 = -6 - 2y_1 - 2y_2 = -6 - 2 \cdot 5 - 2 \cdot (-6) = -4$ . That is,  $\mathbf{y} = (5, -6, -4)^T$ . Thus, the equation  $U\mathbf{x} = \mathbf{y}$  can be written as

$$\begin{pmatrix} 5 & 4 & 3 \\ 0 & 4 & 1 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5 \\ -6 \\ -4 \end{pmatrix},$$

or else

$$5x_1 + 4x_2 + 3x_3 = 5$$
  

$$4x_2 + x_3 = -6$$
  

$$2x_3 = -4$$

This can easily be solved by what is called back substitution. From the third equation, we have  $x_3 = -2$ . Substituting this into the second equation, we have

$$x_2 = \frac{-6 - x_3}{4} = \frac{-6 - (-2)}{4} = -1$$

Finally, substituting these into the second equation, we obtain

$$x_1 = \frac{5 - 4x_2 - 3x_3}{5} = \frac{5 - 4(-1) - 3(-2)}{5} = 3$$

That is, we have

$$\mathbf{x} = (3, -1, -2)^T.$$

3.a) Given

$$A = LU = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} 3 & 6 & 9 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{pmatrix},$$

write A as L'U' such that L' is a lower triangular matrix and U' is an upper triangular matrix such that the elements in the main diagonal of U' are all 1's.

Solution. Write D for the diagonal matrix whose diagonal elements are the same as those of U. That is,

$$D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix}.$$

We have  $A = LU = (LD)(D^{-1}U)$ . Note that the inverse of a diagonal matrix is a diagonal matrix formed by the reciprocal of the elements in the diagonal:

$$D^{-1} = \begin{pmatrix} 1/3 & 0 & 0\\ 0 & 1/2 & 0\\ 0 & 0 & 1/5 \end{pmatrix}.$$

Now, we have

$$LD = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 \\ 3 & 2 & 0 \\ 9 & 4 & 5 \end{pmatrix}.$$

Furthermore,

$$D^{-1}U = \begin{pmatrix} 1/3 & 0 & 0\\ 0 & 1/2 & 0\\ 0 & 0 & 1/5 \end{pmatrix} \begin{pmatrix} 3 & 6 & 9\\ 0 & 2 & 6\\ 0 & 0 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3\\ 0 & 1 & 3\\ 0 & 0 & 1 \end{pmatrix}$$

Thus,

$$A = \begin{pmatrix} 3 & 0 & 0 \\ 3 & 2 & 0 \\ 9 & 4 & 5 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}.$$

b) Explain briefly why pivoting works smoothly with the Crout algorithm (make a comparison with the Dolittle algorithm, for which pivoting does not work smoothly). Do not write the formulas used to calculate the matrix elements in these algorithms, only give a brief explanation using your own words.

**Solution.** When one converts an LU factorization LU with L being an lower diagonal unit matrix into and a factorization L'U' with U' an upper diagonal unit matrix, the diagonal elements of L' are the same as those of U. When doing Gaussian elimination, these diagonal elements are the elements that turned out to be pivot elements. At the *i*th step, the pivot element is chosen from among the elements of column *i* of the matrix  $A^{(i)}$  on or below the diagonal, and only the pivot element among these occurs in the matrix Uobtained at the end of Gaussian elimination, the other elements were changed in the continued course of the algorithms. On the other hand, these elements are the same as the elements of the *i*th column of the matrix L' on or below the diagonal; these elements are calculated at the *i*th step of the Crout algorithm, and are available when one needs to determine the pivot row. In the Doolittle algorithm, these elements are never available, since only the final elements of the matrix are calculated naturally, so only the element actually chosen as pivot element is calculated naturally. The other elements of the *i*th column at the *i*th step need to be calculated tentatively to find the pivot element, and then these calculations will be discarded after the pivot element is found.

(4.a) Change the order of equations in the system

$$-3x + 2y + 7z = 3$$
$$8x - 3y - 2z = -9$$
$$2x + 9y - 3z = 6$$

so that the resulting system can be solved by Gauss-Seidel iteration.

Solution. Moving the first equation to the last place, we obtain the system of equations

$$8x - 3y - 2z = -9$$
$$2x + 9y - 3z = 6$$
$$-3x + 2y + 7z = 3$$

This system of equation is row-diagonally dominant. That is, the absolute value of the each coefficient in the main diagonal is larger than the sum of the absolute values of all the other coefficients on the left-hand side in the same row. Such a system of equations can always be solved by both Jacobi iteration and Gauss-Seidel iteration.

b) Write the equations describing the Gauss-Seidel iteration to solve the system of equations in Part a). Solution. In Gauss-Seidel iteration, one can start with  $x^{(0)} = y^{(0)} = z^{(0)} = 0$ , and for each integer  $k \ge 0$  one can take

$$\begin{aligned} x^{(k+1)} &= \frac{-9 + 3y^{(k)} + 2z^{(k)}}{8} \\ y^{(k+1)} &= \frac{6 - 2x^{(k+1)} + 3z^{(k)}}{9} \\ z^{(k+1)} &= \frac{3 + 3x^{(k+1)} - 2y^{(k+1)}}{7} \end{aligned}$$

c) Write the equations describing Jacobi iteration to solve the system of equations in Part a).

**Solution.** In Jacobi iteration, one can start with  $x^{(0)} = y^{(0)} = z^{(0)} = 0$ , and for each integer  $k \ge 0$  one can take

$$x^{(k+1)} = \frac{-9 + 3y^{(k)} + 2z^{(k)}}{8}$$
$$y^{(k+1)} = \frac{6 - 2x^{(k)} + 3z^{(k)}}{9}$$
$$z^{(k+1)} = \frac{3 + 3x^{(k)} - 2y^{(k)}}{7}$$

5.a) Explain what an orthogonal matrix is.

## Solution.

**Definition.** For a positive integer n, an  $n \times n$  matrix Q is called orthogonal if  $Q^T Q = I$ , where I is the  $n \times n$  identity matrix.

b) Let  $\mathbf{v}$  be an *n*-dimensional column vector such that  $\mathbf{v}^T \mathbf{v} = 1$ , and let *I* be the  $n \times n$  identity matrix. Show that the matrix

$$H = H_{\mathbf{v}} = I - 2\mathbf{v}\mathbf{v}^{T}$$

is orthogonal.

Solution. We have

$$H^T = I^T - 2(\mathbf{v}\mathbf{v}^T)^T = I - 2(\mathbf{v}^T)^T\mathbf{v}^T = I - 2\mathbf{v}\mathbf{v}^T.$$

Furthermore,

$$H^{T}H = HH = (I - 2\mathbf{v}\mathbf{v}^{T})^{2} = I - 2 \cdot 2\mathbf{v}\mathbf{v}^{T} + 4\mathbf{v}\mathbf{v}^{T}\mathbf{v}\mathbf{v}^{T}$$
$$= I - 4\mathbf{v}\mathbf{v}^{T} + 4\mathbf{v}(\mathbf{v}^{T}\mathbf{v})\mathbf{v}^{T} = I - 4\mathbf{v}\mathbf{v}^{T} + 4\mathbf{v}\mathbf{v}^{T} = I;$$

the parenthesis in the third term of the fourth member can be placed anywhere since multiplication of matrices (and vectors) is associative, and the fourth equation holds since  $\mathbf{v}^T \mathbf{v} = 1$ . This shows that H is indeed orthogonal.