All Problems on Prize Exam
Spring 2003
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The secondary source of most of the problems on the prize exams is the Web site
`http://problems.math.umr.edu/index.htm`
(A web site for 20,000 math problems). This site lists many problems but gives no solutions. The primary source for each problem is listed below when available; but even when the source is given, the formulation of the problem may have been changed. Solutions for the problems presented here were obtained without consulting sources for these solutions even when available, and additional information on how to solve these problems might be obtained by consulting the original sources. There was some overlap between the problems on the Junior and Senior prize exams; for this reason, the order the solutions are presented does not quite agree with the order the problems were listed on each of the exams.

1) (JUNIOR 1 and SENIOR 1) Prove that if n is an even integer then

$$\frac{n(n+1)(n+2)}{24}$$

is an integer.

Source. Based on NT-34
Journal: The American Mathematical Monthly
Publisher: Mathematical Association of America
volume(year)page references:
Proposal: 2(1895)241 by R. H. Young
Solution: 3(1896)80 by C. D. Schmitt, J. H. Drummond
Classification: Number Theory, divisibility, products

Solution. One of n, $n+1$, and $n+2$ is divisible by 3, so $n(n+1)(n+2)$ is divisible by 3. We need to show that it is also divisible by 8. Let $n = 2k$, where k is an integer. Then k of $k+1$ is even, so $k(k+1)$ is divisible by to. Hence

$$n(n+2) = 4k(k+1)$$

is divisible by 8, as we wanted to show.

2) (JUNIOR 2 and SENIOR 2) How many positive integers, written in the usual decimal notation, have their digits in strictly increasing order? (No leading zeros are allowed: that is, 0469 would not be an acceptable way of writing 469.)

Source. Problem 65.C
Journal: The Mathematical Gazette
Publisher: The Mathematical Association
volume(year)page references:
Proposal: 65(1981)141
Solution: 65(1981)297 by R. Wakefield
Classification: Combinatorics, digit problems

Solution. Since the digits are in increasing order, zero could only be used as the first digit (from the left). However, only positive integers are considered, and leading zeros are not allowed; so the digit 0 cannot be used at all. If one picks a nonempty subset of the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ of all nonzero digits, there is exactly one way of arranging them in increasing order. Thus the number of integers in question is the number of nonempty subsets of the above set; that is, $2^9 - 1 = 511$.

3) (JUNIOR 3 and SENIOR 3) At a party consisting of at least four people, among any four guests there is one who has previously met the other three. Prove that, among any four guests, there is one who has previously met every person at the party. (It is assumed that "having met" is symmetrical; that is, if A has met B then B has met A.)

Source.
Journal: Mathematical Spectrum
Publisher: Applied Probability Trust
volume(year)page references:
Proposal: 13(1981)92
Solution: 14(1982)62
Classification: Combinatorics, mutual acquaintances, 4 people

Solution. In the language of graphs, the assertion to be proved is that if, among any four vertices of an (undirected) graph, there is one that is connected to the other three, then among any four vertices there is one that is connected to every vertex of the graph other than itself (we consider graphs where a vertex is never connected to itself).

Assuming that not every vertex is connected to every other (i.e., that the graph is not complete), let A and B be two distinct vertices that are not connected to each other. Let X be a vertex different from A and B. X must be connected to any vertex Y that is different from A, B, and X; otherwise the four vertices A, B, X, and Y would violate the assumption.

Now, pick two distinct vertices, C, and D, that are different from A and B. Then, by what we said, both C and D must be connected to every vertex of the graph different from A, B, C, and D. By the assumption, one of C and D must also be connected to the the other three vertices among A, B, C, and D. This vertex will be connected to every vertex of the graph (other than itself).

Since any four vertices of the graph must include at least two that are different from A and B, the desired conclusion follows.

4) (JUNIOR 4) Show that, among any ten consecutive positive integers, at least one is relatively prime to all the others.

Source. Problem 7.6
Journal: Mathematical Spectrum
Publisher: Applied Probability Trust
volume(year)page references:
Proposal: 7(1975)67 by B. G. Eke
Solution: 8(1976)34 by Paul Shovlar
Classification: Number Theory, greatest common divisor, consecutive integers

Solution. If d is a divisor of both m and n, then d is also a divisor of $m-n$. If m and n are among these ten consecutive integers, then $m-n<10$; so, to guarantee that m and n are relatively prime, it is enough to ensure that one of them, say n, is not divisible by any of 2, 3, 5, and 7.

To find such an n, first, remove the multiples of 5; this will remove at most two numbers, only one of which can be odd. Similarly, remove the multiples of 7. These two operations will remove at most two odd numbers; originally, there were five odd numbers, so at least three odd numbers will be left over. If both of two odd numbers are divisible by three, their difference must be divisible by 6; so, among these three odd numbers, at most two can be divisible by 3. Removing these, at least one number will be left over that is not divisible by any of 2, 3, 5, and 7. As we pointed out above, this number is relatively prime to all the others.

5) (JUNIOR 5) A square with edge 7 contains 51 points. Prove that at least three of these points lie inside a circle with radius one.

Source. Journal: Summation
Publisher: Association of Teachers of Mathematics of New York City
volume(year)page references:
Proposal: 24(1978/1)26
Solution: 24(1978/3)35
Classification: Geometry, point spacing, containing figures,
circles, squares

Solution. Cover the square with a rectangular grid of spacing of 7/5 (parallel to the sides of the square). This divides the square into 25 non-overlapping squares, each with side 7/5. One of these must contain at least 3 points (since $51>25\cdot 2$). This square fits inside a circle of radius 1, since $7/5<\sqrt{2}$ (because $7^2<5^2\cdot 2$).

6) (JUNIOR 6) Find all solutions of the equation

$$3\cdot 30^x-6\cdot 15^x-3\cdot 6^x+6\cdot 3^x+2\cdot 5^x-10^x+2^x-2=0.$$

Source.
Journal: The Pi Mu Epsilon Journal
Publisher: Pi Mu Epsilon Fraternity
volume(year)page references:
Proposal: 8(1989)684 by Mohammad K. Azarian
Comment by Tom Engelsman, Mathzeus@peoplepc.com,
entered on 12 Dec 2002:
Break down each term into products of 2's, 3's, & 5's
with the appropriate powers of x. Collecting like terms
results in the factored
exponential equation:

$$(3^{(x+1)} - 1)(5^x - 1)(2^x - 2) = 0.$$

This yields to $x = -1$, 0, or 1.

Solution. The hint above outlines the solution. We give some more detail. Write $u = 2^x$, $v = 3^x$, and $w = 5^x$. The equation becomes

$$3uvw - 6vw - 3uv + 6v + 2w - uw + u - 2 = 0.$$

This can be factored as

$$(u - 2)(3v - 1)(w - 1) = 0.$$

This gives $u = 2$, $v = 1/3$, or $w = 1$, that is $x = 1$, $x = -1$, or $x = 0$, respectively.

7) (JUNIOR 7) Prove that three positive numbers can represent the lengths of the altitudes of a triangle if and only if the sum of the reciprocals of any two of the numbers is greater than the reciprocal of the third.

Source.
Journal: The Two Year College Mathematics Journal
Publisher: Mathematical Association of America
volume(year)page references:
Proposal: 5(1974/4)38 by Martin Berman
Solution: 6(1975/4)26 by Steven Kahn
Title: A Triangle Inequality

Solution. If the sides of a triangle are a, b, c, its altitudes are h_a, h_b, h_c, and its area is A, then

$$a = \frac{2A}{h_a}, \quad b = \frac{2A}{h_b}, \quad c = \frac{2A}{h_c}.$$

Hence, for example,

$$\frac{1}{h_a} + \frac{1}{h_b} > \frac{1}{h_c}$$

is equivalent to $a + b > c$. Thus the condition is equivalent to saying that the sum of any two sides is greater than the third; this latter condition is well-known to be equivalent to assertion that there is a triangle with sides a, b, and c.

Thus, if there is a triangle with altitudes h_a, h_b, h_c, then the condition must be satisfied. On the other hand, if the condition is satisfied, then, forming a triangle with sides

$$a' = \frac{1}{h_a}, \quad b' = \frac{1}{h_b}, \quad c' = \frac{1}{h_c},$$

the obtained triangle can be enlarged appropriately so as to have altitudes h_a, h_b, and h_c.

8) (SENIOR 4) Show that there is no positive continuously differentiable function f on $[0, \infty)$ such that

$$f'(x) \geq (f(x))^2$$

for all $x > 0$.

Source. Based on Problem E3331
Journal: The American Mathematical Monthly
Publisher: Mathematical Association of America
volume(year)page references:
Proposal: 96(1989)524 by Walter Rudin

Solution. Assuming such an f exists, we have

$$\left(\frac{1}{f(x)}\right)' = -\frac{f'(x)}{(f(x)^2)} \leq -1.$$

Hence

$$\frac{1}{f(x)} = \frac{1}{f(0)} + \int_0^x \left(\frac{1}{f(t)}\right)' dt \leq \frac{1}{f(0)} - x.$$

The right-hand side here will be negative for large x, a contradiction.

9) (SENIOR 5) Prove that

$$\frac{2}{1} \cdot \frac{4}{3} \cdot \frac{6}{5} \cdot \ldots \cdot \frac{100}{99} > 12.$$

Source. Problem 65.H
Journal: The Mathematical Gazette
Publisher: The Mathematical Association
volume(year)page references:
Proposal: 65(1981)296 by I. M. Etherington
Solution: 66(1982)159 by L. W. Gates
Classification: Number Theory, inequalities, rational expressions, products

Solution. For the product P in question,

$$P=\prod_{n=1}^{50}\frac{2n}{2n-1}=\frac{2}{1}\cdot\frac{4}{3}\cdot\frac{6}{5}\cdot\frac{8}{7}\cdot\prod_{n=5}^{50}\frac{2n}{2n-1}=\frac{128}{35}\cdot\prod_{n=5}^{50}\frac{2n}{2n-1}.$$

The square of the second factor on the right-hand side can easily be estimated:

$$\Big(\prod_{n=5}^{50}\frac{2n}{2n-1}\Big)^2=\prod_{n=5}^{50}\frac{2n}{2n-1}\cdot\frac{2n}{2n-1}>\prod_{n=5}^{50}\frac{2n}{2n-1}\cdot\frac{2n+1}{2n}=\frac{101}{9}>\frac{100}{9}.$$

The first inequality holds because

$$\frac{2n}{2n-1}=1+\frac{1}{2n-1}>1+\frac{1}{2n}=\frac{2n+1}{2n},$$

and the second equality hods because of obvious cancelations. Thus

$$\prod_{n=5}^{50}\frac{2n}{2n-1}>\frac{10}{3}.$$

Hence

$$P>\frac{128}{35}\cdot\frac{10}{3}=\frac{128\cdot 2}{7\cdot 3}=\frac{256}{21}.$$

Since $12\cdot 21=252$, the right-hand side here is greater than 12, establishing the assertion.

10) (SENIOR 6) For a finite set S of integers, denote by $\Pi(S)$ the product of elements of S; if S is the empty set, take $\Pi(S)=1$. Prove that

$$\sum_{S\subset\{1,2,\cdots,n\}}\Pi(S)=(n+1)!.$$

On the left-hand side, $P(S)$ is summed for all the subsets of the set $\{1,2,\cdots,n\}$ (including the empty set and the whole set itself).

Source. Based on Problem 393
Journal: The College Mathematics Journal
Publisher: Mathematical Association of America
volume(year)page references:
Proposal: 20(1989)68 by Ginger Bolton
Classification: Combinatorics — subsets

Solution. Write $F(n)$ for the product above. Clearly, F(1)=2=2! (the two terms of the sum are produced with S the empty set and with $S = \{1\}$). For $n > 1$, we have

$$F(n) = F(n-1) + nF(n-1);$$

the first term on the right-hand side is obtained by summing for those $S \subset \{1, 2, \cdots, n\}$ with $n \notin S$, and the second term, for those with $n \in S$. Thus the result follows by induction on n.

The argument can even be simplified by extending the validity of the equation for $n = 0$, because then the starting step of the induction is simpler. The argument used to establish the result for $n = 1$ is really just the induction step for that n.

11) (SENIOR 7) Give an example of two series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ such that the first series is (conditionally) convergent, the second one is is divergent, $a_n \neq 0$ for every positive integer n, and $\lim_{n\to\infty} b_n/a_n = 1$. (Note: one cannot require the first series here to be absolutely convergent, according to a well-known comparison test.)

Source. Based on Problem 8.3
Journal: Mathematical Spectrum
Publisher: Applied Probability Trust
volume(year)page references:
Proposal: 8(1976)33
Solution: 8(1976)93
Classification: Analysis, series, pairs of series, ratios, limits, convergence

Solution. Pick $a_n = (-1)^{n+1}/n$ and

$$b_n = \begin{cases} -\frac{1}{n} & \text{if } n \text{ is even,} \\ \frac{1}{n} + \frac{1}{n \log n} & \text{if } n \text{ is odd.} \end{cases}$$

Then $\sum_{n=1}^{\infty} a_n$ is a well known convergent alternating series. On the other hand,

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} a_n + \sum_{\substack{n=1 \\ n \text{ is odd}}}^{\infty} \frac{1}{n \log n},$$

and the second series is divergent (by the integral test; the fact that we are only summing for odd n is just a minor annoyance, and it can easily be shown that the divergence when summing for every positive integer n of this series implies its divergence when only summing for odd n). Thus $\sum_{n=1}^{\infty} b_n$ is divergent.