

JUNIOR PRIZE EXAM
SPRING 2003

1) Prove that if n is an even integer then

$$\frac{n(n+1)(n+2)}{24}$$

is an integer.

2) How many positive integers, written in the usual decimal notation, have their digits in strictly increasing order? (No leading zeros are allowed: that is, 0469 would not be an acceptable way of writing 469.)

3) At a party consisting of at least four people, among any four guests there is one who has previously met the other three. Prove that, among any four guests, there is one who has previously met every person at the party. (It is assumed that “having met” is symmetrical; that is, if A has met B then B has met A .)

4) Show that, among any ten consecutive positive integers, at least one is relatively prime to all the others.

5) A square with edge 7 contains 51 points. Prove that at least three of these points lie inside a circle with radius one.

6) Find all solutions of the equation

$$3 \cdot 30^x - 6 \cdot 15^x - 3 \cdot 6^x + 6 \cdot 3^x + 2 \cdot 5^x - 10^x + 2^x - 2 = 0.$$

7) Prove that three positive numbers can represent the lengths of the altitudes of a triangle if and only if the sum of the reciprocals of any two of the numbers is greater than the reciprocal of the third.