Senior Prize Exam
Spring 2003

1) Prove that if n is an even integer then

$$\frac{n(n+1)(n+2)}{24}$$

is an integer.

2) How many positive integers, written in the usual decimal notation, have their digits in strictly increasing order? (No leading zeros are allowed: that is, 0469 would not be an acceptable way of writing 469.)

3) At a party consisting of at least four people, among any four guests there is one who has previously met the other three. Prove that, among any four guests, there is one who has previously met every person at the party. (It is assumed that "having met" is symmetrical; that is, if A has met B then B has met A.)

4) Show that there is no positive continuously differentiable function f on $[0, \infty)$ such that

$$f'(x) \geq (f(x))^2$$

for all $x > 0$.

5) Prove that

$$\frac{2}{1} \cdot \frac{4}{3} \cdot \frac{6}{5} \cdot \ldots \cdot \frac{100}{99} > 12.$$

6) For a finite set S of integers, denote by $\Pi(S)$ the product of elements of S; if S is the empty set, take $\Pi(S) = 1$. Prove that

$$\sum_{S \subset \{1,2,\cdots,n\}} \Pi(S) = (n+1)!.$$

On the left-hand side, $P(S)$ is summed for all the subsets of the set $\{1, 2, \cdots, n\}$ (including the empty set and the whole set itself).

7) Give an example of two series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ such that the first series is (conditionally) convergent, the second one is is divergent, $a_n \neq 0$ for every positive integer n, and $\lim_{n\to\infty} b_n/a_n = 1$. (Note: one cannot require the first series here to be absolutely convergent, according to a well-known comparison test.)