All Problems on Prize Exam
Spring 2004
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The secondary source of some of the problems on the prize exams is the Web site

`http://problems.math.umr.edu/index.htm`

(A web site for 20,000 math problems). This site lists many problems but gives no solutions. The primary source for each problem is listed below when available; but even when the source is given, the formulation of the problem may have been changed. Solutions for the problems presented here were obtained without consulting sources for these solutions even when available, and additional information on how to solve these problems might be obtained by consulting the original sources. There was some overlap between the problems on the Junior and Senior prize exams; the problems common to both exams are listed only once.

1) (JUNIOR 1 and SENIOR 1) Prove that $n^4 + 3n^2 + 2$ is never a square of an integer.

Source: `http://mathschallenge.net/index.php?section=problems&show=true&titleid=imperfect_square_sum`

First Solution: The reason is that $n^4 + 3n^2 + 2$ is always even, but never divisible by 4; so it cannot be a square of an integer.

In fact, $n^4 + 3n^2$ is always divisible by 4. This is clear if n is even. If n is odd, then then $n^2 + 3$ is divisible by 4. Indeed, writing $n = 2k + 1$, we have $n^2 + 3 = 4k^2 + 4k + 4$. So, again, $n^4 + 3n^2 = n^2(n^2 + 3)$ is divisible by 4.

Second solution: We have

$$n^4 + 3n^2 + 2 = (n^2 + 1)(n^2 + 2).$$

The numbers $n^2 + 1$ and $n^2 + 2$ are relatively prime. Their product can be a square only if each of them is a square (since a number is a square if in its prime factorization each prime has an even exponent; since there are no common primes in the prime factorizations of $n^2 + 1$ and $n^2 + 2$, in the prime factorization of each of them the primes must occur with even exponent so that this be true for their product). However, $n^2 + 1$ is a square only in case $n = 0$; $n^2 + 2$ is not a square in this case.

Third solution: If

$$k = n^4 + 3n^2 + 2 = (n^2 + 1)(n^2 + 2),$$

then k must be between $n^2 + 1$ and $n^2 + 2$, showing that k cannot be an integer.

2) (JUNIOR 2 and SENIOR 2) Let n be a positive integer. Prove that $2^n!$ is divisible by 2^{2^n-1}.

Source: `http://mathschallenge.net/index.php?section=problems&show=true&titleid=factorial_divisibility`

Solution: There are $2^n/2 = 2^{n-1}$ even numbers between 1 and 2^n Each of these contributes a factor of 2 to $2^n!$; this shows that $2^n!$ is divisible by $2^{2^{n-1}}$.

There are $2^n/4 = 2^{n-2}$ numbers between 1 and 2^n are divisible by 4. Each of these contributes a factor of 4 to $2^n!$. However, these numbers are also even, and the fact that they contribute a factor of 2 to $2^n!$ has already been counted; what has not been counted is that each of these numbers contribute an additional factor of 2. This shows that $2^n!$ is divisible by $2^{2^{n-1}+2^{n-2}}$.

There are $2^n/8 = 2^{n-3}$ numbers between 1 and 2^n that are divisible by 8. Each of these contributes a new factor of 2 to $2^n!$ that has not yet been considered ($8 = 4 \cdot 2$, but the factor of 4 has already been considered, since these numbers are also divisible by 4).

Continuing in this manner, we can see that $2^n!$ is divisible by 2 raised to the power

$$\sum_{k=1}^{n} \frac{2^n}{2^k};$$

the term $2^n/2^k$ here arises by noting that, for each k with $1 \le k \le n$, there are $2^n/2^k$ numbers divisible by 2^k between 1 and 2^n; each of these numbers contributes a factor of 2 to $2^n!$, not counted for smaller values of k. It is also easy to see that in this way all factors of 2 of $2^n!$ are counted; that is, $2^n!$ is not divisible by a higher power of 2. The above sum can be evaluated as

$$\sum_{k=1}^{n} 2^{n-k} = \sum_{l=0}^{n-1} 2^l = 2^n - 1.$$

That is, $2^n!$ is divisible by 2^{2^n-1}, and it is not divisible by any higher power of 2.

This argument can be generalized. In fact, Legendre noted that if p is a prime and n is a positive integer, then the highest power of p that divides $n!$ is p raised to the power

$$\sum_{k=1}^{\infty} \left[\frac{n}{p^k} \right],$$

where $[x]$ denotes the integer part of (the largest integer not exceeding) x. The sum on the right-hand side is, of course, a finite sum, since all terms are zero when k is so large that $p^k > n$.

3) (JUNIOR 3 and SENIOR 3) Draw a circle of radius 1 at each of the four vertices of the unit square. Determine the area of the region that is covered by all four circles.

Source: `http://mathschallenge.net/index.php?section=problems&show=`

true&titleid=quarter%20circles

Solution: Inside the unit square, there will be regions covered by exactly two circles, by exactly three circles, and there will be one region covered by exactly four circles. The area of this last region is to be determined. Write A for this area.

Let the unit square in consideration be the square with vertices $(0,0)$, $(0,1)$, $(1,1)$, $(1,0)$. A typical region covered by exactly two circles is the region next to the side connecting the points $(1,0)$ and $(1,1)$ that is outside the circles centered at $(0,0)$ and $(1,1)$. The area of this region is

$$B = 2\int_0^{1/2} \left(1 - \sqrt{1-x^2}\right)\,dx.$$

Using the substitution $x = \sin t$, which gives $dx = \cos t\,dt$, we obtain

$$\int_0^{1/2} \sqrt{1-x^2}\,dx = \int_0^{\pi/6} \cos^2 t\,dt = \int_0^{\pi/6} \frac{1+\cos 2t}{2}\,dt$$
$$= \left[\frac{t}{2} + \frac{\sin 2t}{4}\right]_0^{\pi/6} = \frac{\pi}{12} + \frac{\sqrt{3}}{8}.$$

Thus, the area of the twice covered region under consideration is

$$B = 1 - \pi/6 - \sqrt{3}/4.$$

The four quarter circles cover a total area of π, counting the multiplicity of covering. Adding $4B$, the total area of the four twice covered region, to this, we get $3 + A$, 3 being the total area of the unit square being covered three times, and A is added, since A represents the area of the region that is covered four times. Thus, we have

$$\pi + 4B = 3 + A,$$

i.e.,

$$A = \pi + 4B - 3 = 1 + \frac{\pi}{3} - \sqrt{3} \approx 0.315,147$$

4) (SENIOR 4) Let f and g be non-constant differentiable functions on the real line such that

$$f'(0) = 0, \tag{1}$$

$$f(x+y) = f(x)f(y) - g(x)g(y), \tag{2}$$

and

$$g(x+y) = g(x)f(y) + g(y)f(x). \tag{3}$$

Prove that

$$\big(f(x)\big)^2 + \big(g(x)\big)^2 = 1$$

for all x

Source: `http://www.geocities.com/CapeCanaveral/Lab/4661/Frame_Calculus.html`
Problem 4.

Solution: Proof. According to (2), we have

$$\begin{aligned} f'(x) &= \frac{\partial f(x+y)}{\partial y}\bigg|_{y=0} = f(x)f'(y)\Big|_{y=0} - g(x)g'(y)\Big|_{y=0} \\ &= f(x)f'(0) - g(x)g'(0) = -g(x)g'(0), \end{aligned} \tag{4}$$

where we used (1) to get the last equality. Similarly, using (3) and (1), we obtain

$$\begin{aligned} g'(x) &= \frac{\partial g(x+y)}{\partial y}\bigg|_{y=0} = g(x)f'(y)\Big|_{y=0} + g'(y)\Big|_{y=0} f(x) = \\ & g(x)f'(0) + g'(0)f(x) = f(x)g'(0). \end{aligned} \tag{5}$$

Hence, writing $f^2(x)$ for $\big(f(x)\big)^2$, and similarly for $g(x)$,

$$\begin{aligned} \frac{d}{dx}\big(f^2(x) + g^2(x)\big) &= 2f(x)f'(x) + 2g(x)g'(x) \\ &= -2f(x)g(x)g'(0) + 2f(x)g(x)g'(0) = 0; \end{aligned}$$

thus $f^2(x) + g^2(x)$ is constant.

(4) and (5) imply that $g'(0) \neq 0$, since otherwise we would have $f'(x) = g'(x) = 0$ for all x according to these formulas, whereas f and f are not constant functions, by assumption. Now, we have

$$0 = f'(0)g(0)g'(0),$$

and so $g(0) = 0$ follows.

According to (2), we have

$$f(0) = f(0+0) = f(0)f(0) - g(0)g(0) = f^2(0) - g^2(0) = f^2(0),$$

where the last equation follows, since $g(0) = 0$, as we just saw. As the only solutions of the equation $z^2 = z$ are $z = 0$ and $z = 1$, this implies that $f(0) = 0$ or 1. The former is impossible, since the assumption $f(0) = 0$ together with the equation $g(0) = 0$ implies, for all x, that

$$f(x) = f(x+0) = f(x)f(0) - g(x)g(0) = 0, \tag{2}$$

according to (2), and

(3) $$g(x) = g(x+0) = g(x)f(0) + g(0)f(x) = 0.$$

according to (3); this is impossible, since we assumed that f and g are not constant.

Thus $f(0) = 1$ and $g(0) = 0$. As we showed that $f^2(x) + g^2(x)$ is constant, we have

$$f^2(x) + g^2(x) = f^2(0) + g^2(0) = 1,$$

which is what we wanted to show.

5) (JUNIOR 4) Assume α and β are acute angles (i.e., $0 < \alpha, \beta < \pi/2$) such that

(1) $$\sin^2 \alpha + \sin^2 \beta = 1.$$

Show that

(2) $$\alpha + \beta = \frac{\pi}{2}.$$

Source: No outside source.

Solution: According to (1), we have

$$\sin^2 \alpha + \sin^2 \beta = \sin^2 \alpha + \cos^2 \alpha,$$

i.e.,

$$\sin^2 \beta = \cos^2 \alpha,$$

that is,

$$\sin \beta = \cos \alpha,$$

since both $\sin \beta$ and $\cos \beta$ are positive. Thus,

$$\sin \beta = \sin \left(\frac{\pi}{2} - \alpha \right).$$

As the function $\sin x$ is one-to-one in the interval $(0, \pi/2)$, this means that

$$\beta = \frac{\pi}{2} - \alpha,$$

which is what we wanted to show.

6) (JUNIOR 5) Assume the function f on the real line satisfies the equation

(1) $$f(x)f(y) = f(x+y)$$

for all reals x and y. Assume, further, that f is differentiable at 0. Show that f is differentiable everywhere.

Source: `http://www.geocities.com/CapeCanaveral/Lab/4661/Calculus-2.html`
Problem 7

Solution: Using (1), we have

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{f(x+h)-f(x+0)}{h} \\ &= \lim_{h\to 0} \frac{f(x)f(h)-f(x)f(0)}{h} = f(x) \lim_{h\to 0} \frac{f(h)-f(0)}{h} = f(x)f'(0). \end{aligned}$$

7) (SENIOR 5) Assume α and β are acute angles (i.e., $0 < \alpha, \beta < \pi/2$) such that

$$\sin^2\alpha + \sin^2\beta = \sin(\alpha+\beta). \tag{1}$$

Show that

$$\alpha + \beta = \frac{\pi}{2}. \tag{2}$$

Source: `http://www.geocities.com/CapeCanaveral/Lab/4661/Frame_Trigonometry.html`
Problem 1.

Solution: Using the addition formula for sin, equation (1) can be rewritten as

$$\sin^2\alpha + \sin^2\beta = \sin\alpha\cos\beta + \cos\alpha\sin\beta,$$

that is,

$$\sin\alpha(\sin\alpha - \cos\beta) + \sin\beta(\sin\beta - \cos\alpha) = 0 \tag{3.}$$

Using the identity

$$\sin x - \sin y = 2\cos\frac{x+y}{2}\cos\frac{x-y}{2},$$

we have write

$$\sin\alpha - \cos\beta = \sin\alpha - \sin\left(\frac{\pi}{2} - \beta\right) = 2\cos\frac{\alpha + \pi/2 - \beta}{2}\sin\frac{\alpha+\beta-\pi/2}{2}.$$

Similarly,

$$\sin\beta - \cos\alpha = \sin\beta - \sin\left(\frac{\pi}{2} - \alpha\right) = 2\cos\frac{\beta + \pi/2 - \alpha}{2}\sin\frac{\alpha+\beta-\pi/2}{2}.$$

Substituting these into (3) and dividing by 2, we obtain

$$\sin\frac{\alpha+\beta-\pi/2}{2}\left(\sin\alpha\cos\frac{\alpha+\pi/2-\beta}{2}+\sin\beta\cos\frac{\beta+\pi/2-\alpha}{2}\right)=0.$$

This equation is certainly satisfied if (2) holds, since then the factor on the left is zero. On the other hand, if (2) does not hold, we have

$$\sin\frac{\alpha+\beta-\pi/2}{2}\neq 0,$$

since α and β are acute angles, so we must have

$$\left|\frac{\alpha+\beta-\pi/2}{2}\right|<\pi.$$

In this case, we can divide the last equation by the factor on the left. We obtain

$$\sin\alpha\cos\frac{\alpha+\pi/2-\beta}{2}+\sin\beta\cos\frac{\beta+\pi/2-\alpha}{2}=0.$$

All four trigonometric functions on the left have positive values; the main reason for this is that

$$\left|\frac{\alpha+\pi/2-\beta}{2}\right|<\frac{\pi}{2}\quad\text{and}\quad\left|\frac{\beta+\pi/2-\alpha}{2}\right|<\frac{\pi}{2}.$$

Thus the last equation cannot hold, showing that (2) must not fail.

8) (JUNIOR 6) Let a_1, a_2, …, a_n be positive numbers. Show that

$$\frac{a_1}{a_2}+\frac{a_2}{a_3}+\ldots+\frac{a_{n-1}}{a_n}+\frac{a_n}{a_1}\geq n.$$

Source: `http://www.geocities.com/CapeCanaveral/Lab/4661/Frame_Inequalities.html`
Problem 5

Solution: Writing $a_{n+1}=a_1$, by the inequality of the arithmetic and geometric means we have

$$\frac{1}{n}\sum_{k=1}^{n}\frac{a_k}{a_{k+1}}\geq\left(\prod_{k=1}^{n}\frac{a_k}{a_{k+1}}\right)^{1/n}=1,$$

establishing the desired inequality.

9) (JUNIOR 7) Let f be a real-valued function on the real line and assume that

$$f(xy+x+y)=f(xy)+f(x)+f(y) \tag{1}$$

for all reals x and y. Show that then

$$f(u+v) = f(u) + f(v) \tag{2}$$

for all reals u v.

Source: `http://www.geocities.com/CapeCanaveral/Lab/4661/Calculus-2.html`

Problem 6

Solution: Using (1) with $y = -x$, we have

$$f(x(-x)) = f(x(-x) + x + (-x)) = f(x(-x)) + f(x) + f(-x),$$

and so $f(x) + (f(-x)) = 0$, i.e., $f(-x) = -f(x)$. In particular, $f(0) = 0$.

Let u and v be arbitrary such that

$$u + v \neq -1, \tag{3}$$

and determine x and y such that

$$\begin{aligned} xy + x + y &= u, \\ -xy + x - y &= v. \end{aligned} \tag{4}$$

Note that these equations are solvable for x and y in view of the assumption (3). Indeed, adding these equations, we obtain

$$2x = u + v,$$

and so we can determine x. Subtracting these equations, we obtain

$$2(x+1)y = u - v.$$

From here we can determine y if $x \neq -1$, which is guaranteed to hold by the preceding equation in view of (3).

Using (1), the first equation in (4) implies

$$f(u) = f(xy + x + y) = f(xy) + f(x) + f(y),$$

and the second equation in (4) implies

$$f(v) = f(x(-y) + x + (-y)) = -f(x(-y)) + f(x) + f(-y) - f(xy) + f(x) - f(y).$$

Adding these equations, we obtain

$$f(u) + f(v) = 2f(x) = 2f\left(\frac{u+v}{2}\right).$$

That is
$$f(u)+f(v)=2f\left(\frac{u+v}{2}\right)$$
unless $u+v=-1$. So, if $t\neq -1$ we have
$$f(t)=f(t)+f(0)=2f\left(\frac{t+0}{2}\right)=2f\left(\frac{t}{2}\right).$$
Substitution this with $t=u+v$ into the preceding equation, this implies
$$f(u)+f(v)=f(u+v)$$
unless $u+v=-1$. This establishes (2) unless $u+v=-1$.

Now, let u and v be such that $u+v=-1$. We have to show that (2) holds also in this case. Choose t arbitrarily in such a way that $t+u+v\neq -1$ and $u+t\neq -1$. Then, using (2) three times while avoiding the exceptional case $u+v=-1$, we obtain
$$f(t)+f(u+v)=f(t+u+v)=f(u+t+v)=f(u+t)+f(v)=f(u)+f(t)+f(v),$$
whence
$$f(u+v)=f(u)+f(v)$$
follows.

10) (SENIOR 6) Show that
$$\ln\frac{101}{100}>\frac{2}{201}.$$
You need to give a rigorous proof; approximate numerical calculations done on a calculator are not accepted.

Source: `http://www.geocities.com/CapeCanaveral/Lab/4661/Frame_Non-numerical.html`
Problem 3

Solution: We need to show that
$$\ln\left(1+\frac{1}{100}\right)>\frac{\frac{1}{100}}{1+\frac{1}{2\cdot 100}},$$
i.e., that
$$\ln(1+x)>\frac{x}{1+\frac{x}{2}}$$

for $x = 1/100$. For any x with $0 < x < 1$ we have

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} > x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}.$$

The equation here is the Taylor expansion of $\ln(1+x)$, and the inequality is about truncating an alternating series saying that the sum an alternating sum is always between the values of two adjacent partial sums.

Similarly, for any x with $0 < x < 2$ we have

$$\frac{1}{1+\frac{x}{2}} = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{2^n} < 1 - \frac{x}{2} + \frac{x^2}{4}.$$

As before, the equation here is a Taylor expansion (or, more simply, the sum formula for a geometric series), and the inequality is about truncating an alternating series. Thus,

$$\frac{x}{1+\frac{x}{2}} < x - \frac{x^2}{2} + \frac{x^3}{4}.$$

Hence it will be enough to show that

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} > x - \frac{x^2}{2} + \frac{x^3}{4},$$

i.e., that

$$\frac{x^3}{3} - \frac{x^4}{4} > \frac{x^3}{4}$$

holds with $x = 1/100$. This inequality can be written as

$$x^3 > 3x^4,$$

or else as $1 > 3x$, and this clearly holds with $x = 1/100$.

11) (SENIOR 7) At a theater office, there is a line of $2n$ persons, n of them have only 10 dollar bills, and n of them have only 20 dollar bills. The ticket costs 10 dollars, and the ticket seller has no change initially. What is the probability that every person can be given the proper change right at the moment when it is his/her turn in the line.

Source: `http://problems.math.umr.edu/index.htm`
Journal: The Mathematical Gazette
Publisher: The Mathematical Association
volume(year)page references:
Proposal: 67(1983)228 by Kee-Wai Lau

Solution: 68(1984)59

Classification: Probability, money problems, making change

Solution: One can reformulate the problem as a problem of walking from the point $(0,0)$ in the coordinate system to the point $(2n,0)$ in such a way that from the point (k,l) one can step to the point $(k+1,l+1)$ (an upward step) or to the point $(k+1,l-1)$ (a downward step). What is the probability that one never has to step below the x axis.

Let p_k be the probability that one can walk to from the point $(0,0)$ to the point $(2k,0)$ without stepping below the x axis. Consider the event E_k that one walks from the point $(0,0)$ to the point $(2n,0)$ in such a way that one steps below the x axis that one reaches the point $(2k,0)$ without stepping below the x axis, but the next step is a downward step (i.e., one first gets below the x axis at the point $(2k+1,-1)$. The probability of E_k is $p_k/2$, since after reaching the point $(2k,0)$, the probability of a downward step is $1/2$.

The events E_k are clearly disjoint. The union of all events E_k for $0 \le k \le n-1$ is the event that the walk does get below the x axis; so its complement is the event that one never steps below the x axis. This consideration gives the recursive formula

$$p_n = 1 - \frac{1}{2}\sum_{k=0}^{n-1} p_k.$$

Furthermore, we have $p_0 = 1$.

It follows from the above recursive formula together with the initial condition $p_0 = 1$ that $p_n = 2^{-n}$. Indeed, this is true for $n = 0$. Assuming that we have $p_k = 2^{-k}$ for $k < n$, we obtain

$$p_n = 1 - \frac{1}{2}\sum_{k=0}^{n-1} 2^{-k} = 1 - \frac{1}{2} \cdot \frac{2^{-n}-1}{2^{-1}-1} = 1 - (1 - 2^{-n}) = 2^{-n},$$

which is what we wanted to show.