Junior Prize Exam
Spring 2004

1) Prove that $n^4 + 3n^2 + 2$ is never a square of an integer.

2) Let n be a positive integer. Prove that $2^n!$ is divisible by $2^{2^n - 1}$.

3) Draw a circle of radius 1 at each of the four vertices of the unit square. Determine the area of the region that is covered by all four circles.

4) Assume α and β are acute angles (i.e., $0 < \alpha, \beta < \pi/2$) such that

$$\sin^2 \alpha + \sin^2 \beta = 1. \tag{1}$$

Show that

$$\alpha + \beta = \frac{\pi}{2}. \tag{2}$$

5) Assume the function f on the real line satisfies the equation

$$f(x)f(y) = f(x+y) \tag{1}$$

for all reals x and y. Assume, further, that f is differentiable at 0. Show that f is differentiable everywhere.

6) Let $a_1, a_2, \ldots, a_n$ be positive numbers. Show that

$$\frac{a_1}{a_2} + \frac{a_2}{a_3} + \ldots + \frac{a_{n-1}}{a_n} + \frac{a_n}{a_1} \geq n.$$

7) Let f be a real-valued function on the real line and assume that

$$f(xy + x + y) = f(xy) + f(x) + f(y) \tag{1}$$

for all reals x and y. Show that then

$$f(u+v) = f(u) + f(v) \tag{2}$$

for all reals u v.