Senior Prize Exam
Spring 2004

1) Prove that $n^4 + 3n^2 + 2$ is never a square of an integer.

2) Let n be a positive integer. Prove that $2^n!$ is divisible by $2^{2^n - 1}$.

3) Draw a circle of radius 1 at each of the four vertices of the unit square. Determine the area of the region that is covered by all four circles.

4) Let f and g be non-constant differentiable functions on the real line such that

$$f'(0) = 0, \tag{1}$$

$$f(x+y) = f(x)f(y) - g(x)g(y), \tag{2}$$

and

$$g(x+y) = g(x)f(y) + g(y)f(x). \tag{3}$$

Prove that

$$\bigl(f(x)\bigr)^2 + \bigl(g(x)\bigr)^2 = 1$$

for all x

5) Assume α and β are acute angles (i.e., $0 < \alpha, \beta < \pi/2$) such that

$$\sin^2 \alpha + \sin^2 \beta = \sin(\alpha + \beta). \tag{1}$$

Show that

$$\alpha + \beta = \frac{\pi}{2}. \tag{2}$$

6) Show that

$$\ln \frac{101}{100} > \frac{2}{201}.$$

You need to give a rigorous proof; approximate numerical calculations done on a calculator are not accepted.

7) At a theater office, there is a line of $2n$ persons, n of them have only 10 dollar bills, and n of them have only 20 dollar bills. The ticket costs 10 dollars, and the ticket seller has no change initially. What is the probability that every person can be given the proper change right at the moment when it is his/her turn in the line.