Junior Prize Exam
Spring 2005

1) List the digits that can occur as the last digit of the fourth power of an integer written in the decimal system.

Solution: The possible digits are the last digits of $0^4 = 0$, $1^4 = 1$, $2^4 = 16$, $3^4 = 81$, $4^4 = 256$, $5^4 = 625$, $6^4 = 1296$, $7^4 = 2041$, $8^4 = 4096$, and $9^4 = 6561$ (of course, to determine the last digit, there is no need to calculate all digits of these powers). That is, the possible last digits are 0, 1, 5, and 6.

Note. Instead of doing actual calculations, one can refer to Fermat's theorem. Namely, if 5 is not a divisor of n, then, according to Fermat's Little Theorem, $n^4 \equiv 1 \mod 5$. This allows the last digit of n^4 to be 1 and 6. If 5 is a divisor of n, then 5 is also a divisor of n^4; this allows the last digit of n^4 to be either 0 or 5. This argument shows that the last digit of n^4 can only be 0, 1, 5, and 6. To complete the solution of the problem, one stilll has to show that these numbers can actually occur as the last digit of n^4 by giving examples: $0^4 = 0$, $1^4 = 1$, $2^4 = 16$, and $5^4 = 625$,

2) Let f be a function defined for all real numbers x such that

$$f(x+1) + f(x-1) = \sqrt{2} f(x) \tag{1}$$

holds for all real x. Prove that f is periodic.

Source: A problem in the Rózsa Péter Memorial Competition in Hungary, 2001. See Középiskolai Matematikai és Fizikai Lapok (Mathematics and Physics Journal for Secondary Schools), Budapest (Hungary), Vol. 53 (January 2003), p. 16.

Solution: Multiplying the above equation by $\sqrt{2}$, we obtain

$$\sqrt{2} f(x+1) + \sqrt{2} f(x-1) = 2f(x). \tag{2}$$

Noting that equation (1) implies with $x+1$ and $x-1$, respectively, replacing x that

$$\sqrt{2} f(x+1) = f(x+2) + f(x) \tag{3}$$

and

$$\sqrt{2} f(x-1) = f(x) + f(x-2),$$

and substituting these into (2), we obtain

$$f(x+2) + 2f(x) + f(x-2) = 2f(x),$$

i.e.,

$$f(x+2) = -f(x-2).$$

This is valid for every real x; that is, we have $f(x+4) = -f(x)$ and $f(x+8) = f(x+4)$. That is, $f(x+8) = f(x)$ for all x, showing that f is indeed periodic.

A deeper understanding of this problem can be gained if one considers the forward shift operator E, acting on functions defined for reals (that is, given such a function f, Ef is another such function, defined by the equation

$$(Ef)(x) = f(x+1) \qquad \text{for all real } x.$$

One usually omits the parentheses, and instead of $(Ef)(x)$ one writes $Ef(x)$. One can add the definitions $E^0 f = f$, and $E^{k+1}f = E(E^k)f$.[1] With this equation (2) can be written as

$$\sqrt{2}Ef = f + E^2 f,$$

or as

$$(E^2 - \sqrt{2}E + 1)f = 0. \tag{4}$$

Considering this as the equation $E^2 - 2E + 1 = 0$, one obtains the solutions

$$E = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i \quad \text{and} \quad E = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i,$$

where $i = \sqrt{-1}$. The numbers on the right-hand side are eighth roots of unity; i.e., for these values of E, we have $E^8 = 1$. That is, this formal argument shows that we have $E^8 f = f$, or else $f(x+8) = f(x)$ for all x.

This argument is purely formal, yet it can be turned into a rigorous proof. Namely, the fact that the zeros of the polynomial $X^2 - 2X + 1$ are eighth roots of unity means that this polynomial divides the polynomial $X^8 - 1$. In fact,

$$(X^4 - 1)(X^2 + \sqrt{2}X + 1)(X^2 - \sqrt{2}X + 1) = X^8 - 1.$$

Since calculations with E obey the rules of a polynomial ring over the reals, this means that

$$(E^4 - 1)(E^2 + \sqrt{2}E + 1)(E^2 - \sqrt{2}E + 1)f = (E^8 - 1)f.$$

The left-hand side here is 0 according to (4), showing (now rigorously) that $(E^8 - 1)f = 0$, i.e., that $E^8 f = f$, or else that $f(x+8) = f(x)$ for all x.

3) Assume f is a function such that (i) f continuous for $x \geq 0$, (ii) f is differentiable for $x > 0$, (iii) $f(0) = 0$, and (iv) f' is increasing for $x > 0$. Write

$$g(x) = \frac{f(x)}{x} \qquad \text{for} \quad x > 0.$$

[1] In fact, one can write $E^{-1}f(x) = f(x-1)$, and then equation (1) can be written as $(E+E^{-1})f = \sqrt{2}f$. However, the rigorous justification of the arguments below is simpler if one stays within the polynomial ring $\mathbb{R}[E]$, where $\mathbb{R}$ is the field of reals; therefore, it is easier to avoid considering negative powers of E.

Show that g is increasing for $x > 0$.

Source: The problem is given as Exercise 6 on p. 114 in Walter Rudin, *Principles of Mathematical Analysis*, third edition, McGraw Hill, New York 1976.

Solution: We have

$$g'(x) = \left(\frac{f(x)}{x}\right)' = \frac{f'(x)x - f(x)}{x^2}.$$

It will be sufficient to show that $g'(x) \geq 0$, i.e., that $f(x) \leq f'(x)x$, holds for $x > 0$. This inequality indeed holds, since we have

$$f(x) = f(x) - f(0) = f'(\xi)(x - 0) = f(\xi)x \leq f'(x)x$$

for some ξ with $0 < \xi < x$, the Mean-Value Theorem of differentiation justifying the second equality; the inequality here holds in view of the assumption that f' is increasing.

4) How many subsets not containing consecutive integres does the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ have. Include the empty set in your count. (Two integers are called consecutive if their difference is 1.)

Source: The problem is related to a well-known fact about the Lucas numbers, defined by $L_1 = 1$, $L_2 = 3$, and $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$. The details are given below.

Solution: Write $[1, n]$ for the set $\{1, 2, \dots, n\}$, and write C_n for the number of subsets of $[1, n]$ not containing consecutive integers. We have $C_1 = 2$; indeed, the subsets of $[1, 1] = \{1\}$ that qualify are the empty sets and the set $\{1\}$. It is also easy to see that $C_2 = 3$; indeed, the subsets of $[1, 2]$ that qualify are the empty set, the set $\{1\}$, and the set $\{2\}$.

Further, it is easy to see that $C_n = C_{n-1} + C_{n-2}$ for $n \geq 2$. Indeed, given an integer $n \geq 2$ and a set $X \subset [1, n]$ such that X does not contain consecutive integers, then either $n \notin X$, in which case $X \subset [1, n-1]$, or $n \in X$, in which case $X \setminus \{n\} \subset [1, n-2]$. The number of sets X satisfying the former condition is C_{n-1}, and the number of those satisfying the latter condition is the same as the number of diffrent sets $X \setminus \{n\}$, i.e., C_{n-2}. Thus, indeed, $C_n = C_{n-1} + C_{n-2}$, as claimed.

Using these observations, we have $C_3 = C_1 + C_2 = 2 + 3 = 5$, $C_4 = C_2 + C_3 = 3 + 5 = 8$, $C_5 = C_3 + C_4 = 5 + 8 = 13$, $C_6 = C_4 + C_5 = 8 + 13 = 21$, $C_7 = C_5 + C_6 = 13 + 21 = 34$, $C_8 = C_6 + C_7 = 21 + 34 = 55$, $C_9 = C_7 + C_8 = 34 + 55 = 89$, $C_{10} = C_8 + C_9 = 55 + 89 = 144$. That is, there are 144 subsets satisfying the requirements.

The following related problem leads to the well-known Lucas numbers (see below). Modify the above question as follows:

How many subsets not containing cyclically consecutive integres does the set $[1, n]$ have. if the numbers 1 and n are also considered consecutive. Include the empty set in your count. (For $n = 1$ this means that 1 is consecutive to itself,

Writing L_n for the number of subsets as described, it is easy to see that $L_n = C_n - C_{n-4}$ for $n \geq 5$. Indeed, from the set of subsets of $[1, n]$ considered before (i.e., those subsets not containing consecutive integers, with 1 and n not counted as consecutive unless $n = 2$) we now have to discard those sets X that contain both 1 and n. In this case $X \setminus \{1, n\}$ is a subset of the interval $[3, n-2]$ not containing two consecutive integers; the number of these is C_{n-4}, verifying the above formula. This formula can be extended also to $n = 1$, 2,, 3, 4 by taking $C_{-3} = 1$, $C_{-2} = 0$, $C_{-1} = 1$, and $C_0 = 1$. Indeed, when counting the subsets of $[1, n]$ for $n = 1$, 2, 3, 4, we have to discard the set $\{1\}$ in case $n = 1$ (this is because, somewhat unnaturally, in this case 1 is considered consecutive to itself), in case $n = 2$ no subset needs to be discarded, in cae $n = 3$ the set $\{1, 3\}$ needs to be discarded, and in case $n = 4$ the set $\{1, 4\}$ needs to be discarded.

Note that the relation $C_n = C_{n-1} + C_{n-2}$, established for $n \geq 3$, now extends to the values of $n = -1$, 0, 1, and 2 as well. Thus, for $n \geq 3$ we have $L_n = C_n - C_{n-4} = (C_{n-1} + C_{n-2}) - (C_{n-5} + C_{n-6}) = (C_{n-1} - C_{n-5}) + (C_{n-2} - C_{n-6}) = L_{n-1} + L_{n-2}$, i.,e,

$$L_n = L_{n-1} + L_{n-2}.$$

This recursive relation together with the initial values $L_1 = 1$ and $L_2 = 3$ defines the Lucas numbers. See

`http://mathworld.wolfram.com/LucasNumber.html`

5) Given an arbitrary quadrilateral, erect a square looking outward on each side. The centers of these squares form a new quadrilateral. Show that the diagonals of this new quadrilateral are perpendicular and have the same length.

Hint. A proof using complex numbers is probably simpler than a direct geometric proof.

Source: The problem is given as Problem 2 on p. 54 in Tibor Szele, *Bevezetés az Algebrába* (Introduction to Algebra. In Hungarian), third edition Tankönyvkiadó, Budapest (Hungary), 1963.

Solution: Let the complex numbers a_1, a_2, a_3, and a_4 represent the vertices of the given quadrilateral, going around counterclockwise. To simplify the calculation, define a_n for every integer n (positive, negative, or zero) by putting $a_n = a_k$ if $n \equiv i \mod 4$ ($k = 1$, 2, 3, or 4). For the vertex b_n of the new quadrilateral that is the center of the square erected over the side connecting a_n and a_{n+1} we have

$$b_n = a_{n+1} + (a_n - a_{n+1})\frac{1+i}{2} = a_n\frac{1+i}{2} + a_{n+1}\frac{1-i}{2},$$

where $i = \sqrt{-1}$. For the diagonal vector $b_{n+2} - b_n$ of the new quadrilateral we have

$$b_{n+2} - b_n = (a_{n+2} - a_n)\frac{1+i}{2} + (a_{n+3} - a_{n+1})\frac{1-i}{2}.$$

Replacing n with $n + 1$ here, we obtain the expression for the other diagonal, that is, the diagonal vector $b_{n+3} - b_{n+1}$:

$$\begin{aligned} b_{n+3} - b_{n+1} &= (a_{n+3} - a_{n+1})\frac{1+i}{2} + (a_{n+4} - a_{n+2})\frac{1-i}{2} \\ &= (a_{n+3} - a_{n+1})\frac{1+i}{2} + (a_{n+2} - a_n)\frac{-1+i}{2}, \end{aligned}$$

where the second equation was obtained by noting that $a_{n+4} = a_n$. Comparing the right-hand sides of the last two displayed equations, it is clear that $(b_{n+2} - b_n)i = b_{n+3} - b_{n+1}$, showing that the vectors $b_{n+2} - b_n$ and $b_{n+3} - b_{n+1}$ have equal length and are perpendicular, as claimed.

6) Let α, β, and γ be the angles of a triangle. Show that

$$\cos\alpha + \cos\beta + \cos\gamma < 2.$$

Source: Problem 170 on p. 340 in Emil Molnár, *Matematikai Versenyfeladatok Gyüjteménye, 1947–1970* (Collection of Competition Problems in Mathematics, 1947–1970. In Hungarian), Tankönyvkiadó, Budapest (Hungary), 1974.

Solution: We are going to prove the stronger inequality that

$$\cos\alpha + \cos\beta + \cos\gamma \le 3/2, \tag{1}$$

with equality holding only in case $\alpha = \beta = \gamma = \pi/3$. First, if $\alpha > 2\pi/3$ then the inequality is clearly true, since then $\cos\alpha < -1/2$, and so

$$\cos\alpha + \cos\beta + \cos\gamma < -1/2 + 1 + 1 = 3/2.$$

So we may assume that $0 < \alpha \le 2\pi/3$, $0 < \beta \le 2\pi/3$, and $0 < \gamma \le 2\pi/3$. Note that for an x with $0 \le x \le 2\pi/3$ we have

$$\cos x - \cos\frac{\pi}{3} \le -\left(x - \frac{\pi}{3}\right)\sin\frac{\pi}{3}; \tag{2}$$

with equality holding only in case $x = \pi/3$. This inequality simply says that the graph of $\cos x$ is below the tangent line at $\pi/3$ in the interval $(0, 2\pi/3)$. This is obvious by inspecting the graph of $\cos x$. We will verify this inequality formally below, but first we point out how to derive (1) with the aid of this inequality. Using this inequality with α, β, and γ replacing x, and observing that $\cos\pi/3 = 1/2$, we have

$$\begin{aligned} cos\alpha + \cos\beta + \cos\gamma - \frac{3}{2} &= \left(\cos\alpha - \cos\frac{\pi}{3}\right) + \left(\cos\beta - \cos\frac{\pi}{3}\right) + \left(\cos\gamma - \cos\frac{\pi}{3}\right) \\ &\le -\left(\alpha - \frac{\pi}{3}\right)\sin\frac{\pi}{3} - \left(\beta - \frac{\pi}{3}\right)\sin\frac{\pi}{3} - \left(\gamma - \frac{\pi}{3}\right)\sin\frac{\pi}{3} = -(\alpha+\beta+\gamma-\pi)\sin\frac{\pi}{3} = 0; \end{aligned}$$

the inequality here follows according to (2), and the last equality follows since we have $\alpha + \beta + \gamma = \pi$ for the angles of a triangle.

To establish inequality (2), consider the function

$$f(x) = \cos x - \cos\frac{\pi}{3} + \left(x - \frac{\pi}{3}\right)\sin\frac{\pi}{3};$$

We have

$$f'(x) = \sin\frac{\pi}{3} - \sin x.$$

In the interval $(0, \pi/3)$ we have $f'(x) > 0$, so f is increasing in this interval. In the interval $(\pi/3, 2\pi/3)$ we have $f'(x) < 0$, so f is decreasing there. Hence f assumes its maximum in the interval $[0, 2\pi/3]$ at $x = \pi/3$ (the endpoints of the interval can be included in view of the continuity of f). At $x = \pi/3$ we have $f(x) = 0$; so, everywhere else in the interval $[0, 2\pi/3]$ we have $f(x) < 0$. This establishes inequality (2).

Note. The above solution exploited the concavity of the function $y = \cos x$.[2] Convexity and concavity is frequently exploited in mathematics. Jensen's remarkable inequality is based on convexity, and is discussed in many advanced texts on mathematical analysis. See e.g. A. Zygmund, *Trigonometric Series*, Vol. I and II, Second Edition, Cambridge University Press, London–New York–Melbourne, 1959 and 1977, pp. 21–26.

In the above argument, we had to go somewhat beyond the simple concavity of the function $y = \cos x$ in that we had to establish inequality (2) beyond the range of concavity.

There are various other proofs of inequality (2) that, using some ingenuity, avoid the differential calculus needed in establishing (2); these proofs, however, do not highlight the key role played by convexity in the result.

7) Let a and b be positive integers. Show that there can only be finitely many positive integers n for which both $an^2 + b$ and $a(n+1)^2 + b$ are squares of integers.

Source: Problem 2 on p. 8 in János Surány (editor), *Matematikai Versenytélek, IV. rész* (Competition Problems in Mathematics, Part IV. In Hungarian), TypoTEX, Budapest (Hungary), 1998.

Solution: Assume, on the contrary, that

$$(1) \qquad k^2(n) = an^2 + b \quad \text{and} \quad l^2(n) = a(n+1)^2 + b$$

hold for $n \in S$, where S is an infinite set of positive integers, and $k(n)$ and $l(n)$ are positive integers; of course, $k^2(n)$ abbreviates $\big(k(n)\big)^2$; similarly for $l^2(n)$. We have

$$\begin{aligned}\lim_{\substack{n\to\infty\\ n\in S}} \big(l(n) - k(n)\big) &= \lim_{\substack{n\to\infty\\ n\in S}} \frac{l^2(n) - k^2(n)}{l(n) + k(n)} = \lim_{\substack{n\to\infty\\ n\in S}} \frac{a(2n+1)}{\sqrt{a(n+1)^2 + b} + \sqrt{an^2 + b}} \\ &= \lim_{\substack{n\to\infty\\ n\in S}} \frac{2a + \frac{a}{2n}}{\sqrt{a\left(1 + \frac{1}{n}\right)^2 + \frac{b}{n^2}} + \sqrt{a + \frac{b}{n^2}}} = \frac{2a}{\sqrt{a} + \sqrt{a}} = \sqrt{a}.\end{aligned}$$

The sequence of the left-hand side assumes only integer values. For it to have a limit, it must be eventually constant. That is, we have

$$l(n) - k(n) = \sqrt{a}$$

[2] A function that is concave downward is called *concave*, and a function that is concave upward is called *convex*. The designations *concave upward* and *concave downward* are not commonly used in mathematics outside introductory calculus courses.

for large enough $n \in S$. This means that $\sqrt{a}$ must be an integer; write $a = c^2$, where c is a positive integer.

We have $k^2(n) = (cn)^2 + b$ for every $n \in S$ according to (1); hence $k(n) \geq cn + 1$, as $b > 0$. That is, we have

$$k^2(n) = (cn)^2 + b \geq (cn + 1)^2 = (cn)^2 + 2cn + 1,$$

for $n \in S$. i.e.,

$$b \geq 2cn + 1,$$

or else

$$n \leq \frac{b-1}{2c} \tag{2,}$$

for $n \in S$. This inequality shows that the set S is finite. This contradicts our assumption that S is infinite. This contradiction proves the assertion that (1) can hold only for finitely many values of n.

Note. We need to invoke a contradiction, and cannot directly say that this inequality accomplishes the proof that S is finite. Namely, this inequality depends on the statement $\sqrt{a}$ is an integer, and that statement depends on the assumption that S is infinite. Thus, this inequality, being hypothetical rather than describing the actual state of affairs, does not in itself show anything. In particular, if (1) holds only for finitely many integers n, the inequality

$$n \leq \frac{b-1}{2\sqrt{a}},$$

identical to (2) if we write $\sqrt{a}$ (which need not be an integer now) instead of c, need not be satisfied for such n.

Soon after the exam, solutions will appear on the Web site
`http://www.sci.brooklyn.cuny.edu/~mate/prize05/index.html`