Junior Prize Exam
Spring 2005

1) List the digits that can occur as the last digit of the fourth power of an integer written in the decimal system.

2) Let f be a function defined for all real numbers x such that

$$f(x+1)+f(x-1)=\sqrt{2}f(x) \tag{1}$$

holds for all real x. Prove that f is periodic.

3) Assume f is a function such that (i) f continuous for $x \ge 0$, (ii) f is differentiable for $x > 0$, (iii) $f(0) = 0$, and (iv) f' is increasing for $x > 0$. Write

$$g(x) = \frac{f(x)}{x} \qquad \text{for} \quad x > 0.$$

Show that g is increasing for $x > 0$.

4) How many subsets not containing consecutive integres does the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ have. Include the empty set in your count. (Two integers are called consecutive if their difference is 1.)

5) Given an arbitrary quadrilateral, erect a square looking outward on each side. The centers of these squares form a new quadrilateral. Show that the diagonals of this new quadrilateral are perpendicular and have the same length.

Hint. A proof using complex numbers is probably simpler than a direct geometric proof.

6) Let α, β, and γ be the angles of a triangle. Show that

$$\cos\alpha + \cos\beta + \cos\gamma < 2.$$

7) Let a and b be positive integers. Show that there can only be finitely many positive integers n for which both $an^2 + b$ and $a(n+1)^2 + b$ are squares of integers.

Soon after the exam, solutions will appear on the Web site
`http://www.sci.brooklyn.cuny.edu/~mate/prize05/index.html`