All Problems on Prize Exam
Spring 2006
Version Date: Tue Jan 18 15:20:16 EST 2005

The source for each problem is listed below when available; but even when the source is given, the formulation of the problem may have been changed. Solutions for the problems presented here were obtained without consulting sources for these solutions even when available, and additional information on how to solve these problems might be obtained by consulting the original sources. There was some overlap between the problems on the Junior and Senior prize exams; the problems common to both exams are listed only once.

1) (JUNIOR 1 and SENIOR 1) Given a positive integer n, show that $10n^3 + 3n^2 - n$ is divisible by 6.

Source: Középiskolai Mathematikai Lapok (The Hungarian Mathematics Journal for High Schools), Problem 79 (1925/10). See

`http://www.sulinet.hu/komal/`

Proposed by Ferenc Kárteszi. In Hungarian. The Web site also provides also has an English translation, see

`http://www.komal.hu/info/bemutatkozas.e.shtml`

Solution: We have

$$10n^3 + 3n^2 - n = 2(5n^3 + n^2) + n(n-1);$$

since either n or $n-1$ is divisible by 2, the left-hand side also must be divisible by 2. Further, we have

$$10n^3 + 3n^2 - n = 3(3n^3 + n^2) + n(n-1)(n+1);$$

since one of n, $n-1$, and $n+1$ is divisible by 3, the left-hand side must be divisible by 3. Thus $10n^3 + 3n^2 - n$ is divisible by both 2 and 3, and so it is divisible by 6.

2) (JUNIOR 2 and SENIOR 2) A function f defined on the interval $[0, 1]$ satisfies $f(0) = f(1)$ and is such that for any x and y with and $0 \le x < y \le 1$ we have

$$|f(y) - f(x)| < y - x.$$

Show that we have

$$|f(x) - f(y)| < \frac{1}{2}$$

whenever $0 \le x < y \le 1$.

Source: Chinese Mathematical Olympiad, 1983/1. See

`http://www.problemcorner.org`

Solution: Extend the function f to the interval $[1, 2]$ by putting $f(x+1) = f(x)$ for $x \in (0, 1]$. Observe that the inequality

$$|f(y) - f(x)| < y - x$$

remains valid whenever $0 \le x < y \le 2$. In fact, this is already known if $0 \le x < y \le 1$, and the case when $1 \le x < y \le 2$ is immediate. Finally, in case $0 \le x < 1 < y \le 2$ we have

$$|f(y) - f(x)| \le |f(y) - f(1)| + |f(1) - f(x)| < (y-1) + (1-x) = y - x.$$

Now, given x, y with $0 \leq x < y \leq 1$ we have either $y - x \leq 1/2$ or $(x+1) - y \leq 1/2$. In the former case we have

$$|f(y) - f(x)| < y - x \leq \frac{1}{2},$$

and in the latter case we have

$$|f(y) - f(x)| = |f(x+1) - f(y)| < (x+1) - y \leq \frac{1}{2}.$$

This completes the proof.

3) (JUNIOR 3 and SENIOR 3) Given 5 points in a square with side a, show that two of them are within a distance of at most $a/\sqrt{2}$ of each other.

Source: Problem 9 Delta 6.1-3, proposed by Anthony Biagioli. See

`http://www.problemcorner.org`

Solution: Divide the square into four smaller squares by drawing two lines parallel to the sides through the center of the square. Out of the five points, at least two must fall into (i.e., inside or on the boundary of) one of these smaller squares. The distance of these two points cannot be greater than the length of the diagonal of this smaller square, that is $a/\sqrt{2}$.

4) (JUNIOR 4) Given numbers x_1, x_2, $\dots$ x_n, x_{n+1} such that $x_{n+1} = x_1$, x_i is $+1$ or -1 for each i with $1 \leq i \leq n+1$, and

$$\sum_{i=1}^{n} x_i x_{i+1} = 0,$$

show that n is divisible by 4.

Source: Sándor Róka, *2000 feladat an elemi matematika köréből* (2000 problems in elementary mathematics), Typotex, Budapest (Hungary), 2000, ISBN 963 9132-50-0, Problem 1737, p. 151,

Solution: The product $x_i x_{i+1}$ is either $+1$ or -1; to get zero as the sum of these products, each of these values must occur exactly $n/2$ times (so n is certainly even). When $x_i x_{i+1} = 1$, we have $x_{i+1} = x_i$. When $x_i x_{i+1} = -1$, we have $x_{i+1} = x_i + 2$ or $x_{i+1} = x_i - 2$. As $x_{n+1} = x_1$, we must have $x_{i+1} = x_i + 2$ exactly as many times as $x_{i+1} = x_i - 2$; that is, each of these possibilities must occur $n/4$ times. So $n/4$ must be an integer.

5) (JUNIOR 5) Show that

$$\sum_{i=1}^{n} \frac{1}{i} = 1 + \frac{1}{2} + \ldots + \frac{1}{n}$$

is not an integer for any integer $n \geq 2$.

Source: Brazilian Mathematical Olympiad, 1983/3. See

`http://www.problemcorner.org`

(however, the problem has a much earlier history).

Solution: We have

$$\sum_{i=1}^{n} \frac{1}{i} = \frac{\sum_{i=1}^{n} \frac{n!}{i}}{n!}, \tag{1}$$

where $n! = \prod_{i=1}^{n} i = 1 \cdot 2 \cdot \ldots \cdot n$. In the fraction displayed on the right-hand side, both the numerator and the denominator are integers. We will show that the denominator is divisible by a higher power of 2 than the numerator; this will immediately imply that the fraction is not an integer (since the denominator is then not a divisor of the numerator).

To this end, let k be the integer such that $n < 2^{k+1} \leq 2n$, in which case

$$\frac{n}{2} < 2^k \leq n.$$

It is easy to see that no integer i with $1 \leq i \leq n$ other than 2^k is divisible by 2^k; therefore, for each i with $1 \leq i \leq n$ and $i \neq 2^k$, the integer $n!/i$ is divisible by a higher power of 2 than $n!/2^k$. That is, for integers $x, y \geq 0$ writing $2^x \parallel y$ to mean that y is divisible by 2^x (in symbols: $2^x \mid y$) and y is not divisible by 2^{x+1} (in symbols: $2^{x+1} \nmid y$), let the positive integer l be such that

$$2^l \parallel \frac{n!}{2^k}.$$

Then $2^{l+1} \mid n!/i$ for each i with $1 \leq i \leq n$ and $i \neq 2^k$. Then

$$2^{l+1} \nmid \sum_{i=1}^{n} \frac{n!}{i}$$

(since every term on the right-hand side except the one corresponding to $i = 2^k$ is divisible by 2^{l+1}). On the other hand, $n!$ is certainly divisible by 2^{l+1} (in fact, clearly, $2^{l+k} \parallel n!$, and $k \geq 1$ in view of $n \geq 2$). Hence the fraction on the right-hand side of (1) cannot be an integer.

6) (JUNIOR 6) Suppose $f(x)$ and $g(x)$ are nonzero real polynomials satisfying

$$f(x^2 + x + 1) = g(x)f(x).$$

Show that $f(x)$ has even degree.

Source: Mathematics Magazine 60(1987)40

Solution: Assume f has odd degree. Then f must has at least one real zero (this is because of the Intermediate-Value Theorem, since $f(x)$ will have different signs when $x \to -\infty$ and $x \to +\infty$). Let α be its largest real zero. Then, in view of the equation

$$f(x^2 + x + 1) = g(x)f(x)$$

$\alpha^2 + \alpha + 1$ is also a zero of f. This is, however, a contradiction since $\alpha^2 + \alpha + 1 > \alpha$.

7) (JUNIOR 7) In the complex numbers, the polynomial x^2+y^2 can be factored as $(x+iy)(x-iy)$. Show that the polynomial $x^2 + y^2 + z^2$ cannot be written as a product

$$(ax + by + cz)(Ax + By + Cz),$$

where a, b, c, and A, B, C are complex numbers.

Source. Journal: The AMATYC Review, 63. AMATYC N-3 by Stanley Rabinowitz. See
Publisher: American Mathematical Association of Two-Year Colleges
volume(year)page references:
Proposal: 9(1988/2)71 by Stanley Rabinowitz
Solution: 10(1989/2)68 by Stephen Plett, Joseph Wiener, Stanley
Rabinowitz
Additional solvers listed: 11(1989/1)75
Title: Factoring Sums of Squares
See

`http://www.problemcorner.org`

Solution. Assuming

$$x^2 + y^2 + z^2 = (ax + by + cz)(Ax + By + Cz),$$

we have, for example,

$$Aa = 1, \quad Bb = 1, \quad aC + Ac = 0, \quad \text{and} \quad bC + Bc = 0.$$

Multiplying the third equation by a and the fourth one by b and using the first and second equations, we obtain

$$a^2C + c = 0 \quad \text{and} \quad b^2C + c = 0.$$

Thus $a^2 = b^2$ (since $C \neq 0$, as $Cc = 1$ by an equation similar to the ones stated above). Using similar arguments, we can show that $a^2 = b^2 = c^2$ and $A^2 = B^2 = C^2$. Multiplying a, b, c by $a^{-1/2}$ and A, B, C by $a^{1/2}$, we may assume that $a^2 = b^2 = c^2 = A^2 = B^2 = C^2 = 1$. Then the equation $a^2C + c = 0$ above implies $C = -c$. Similarly, $A = -a$, and $B = -b$.

Hence, the equation $aC + Ac = 0$ becomes $-2ac = 0$. This is, however, impossible, since neither a nor c is zero, because $Aa = 1$ (as stated above) and (similarly) $Cc = 1$.

Note. One might imagine that $x^2 + y^2 + z^2$ is the product of more than two linear factors if x, y, z each occur in exactly two of these factors. But, in fact, even this cannot happen:

The polynomial $x^2 + y^2 + z^2$ does not factor as the product of two or more linear factors.

Indeed, assuming that such a factorization exist, we would have the equation

$$x^2 + y^2 + z^2 = (x - P(y, z))(xR(y, z) - Q(y, z)),$$

where $P(y, z)$ and $Q(y, z)$ and $R(y, z)$ are polynomials of y and z. To get this factorization, in the factorization into linear factors, we can take two linear factors containing x, and multiplying the remaining factors into the second one of these (which will imply that $P(y, z)$ must be linear, but this is of no importance at this point). Multiplying out the right-hand side, and considering the two sides as polynomials of x only, the coefficients of x must agree, so we must have $R(y, z) = 1$ for all values of y and z, so we must have $R(y, z) \equiv 1$, where the sign $\equiv$ here means that the two sides agree as polynomials (i.e., they have the same coefficients).[1]

Noting that $R(y, z) \equiv 1$ and multiplying out the right-hand side in the above equation, we have

$$x^2 + y^2 + z^2 = x^2 - (P(y, z) + Q(y, z))x + P(y, z)Q(y, z).$$

For this equation to be true for a all x, the coefficients of x and y must agree, and so we must have

$$P(y, z) + Q(y, z) = 0 \quad \text{and} \quad P(y, z)Q(y, z) = y^2 + z^2.$$

[1]In showing that $R(y, z) = 1$, we can consider $R(y, z)$ as the polynomial only of y and show that all coefficients of this polynomial must be zero except for the constant term.

A similar argument can be used to show that x cannot occur in more than two factors of the product. Indeed, if x occurred in more than two factors, then, after multiplying out and considering the result as a polynomial of x only, the coefficient of the highest power of x in the product will have to be zero. This coefficient is, however, the product of all the factors not containing x. So the product of all the factors not containing x must be zero, which means that the whole product must be zero.

This equation can easily be solved for $P(y,z)$ and $Q(y,z)$:

$$P(y,z) = \pm\sqrt{y^2+z^2} \quad \text{and} \quad Q(y,z) = \mp\sqrt{y^2+z^2}.$$

These equations, however, cannot hold. The simplest reason for this is that the partial derivatives of $P(y,z)$ and $Q(y,z)$ exist at every point, since they are polynomials, while the partial derivatives of $\sqrt{x^2+y^2}$ do not exist at the point $x=0$, $y=0$.

For this last argument, to avoid having to do with a function of two complex variables, one can consider y and z real variables. Indeed, if the above equations cannot hold with real variables y and z, they cannot hold with complex variables y and z, *a fortiori*.[2]

8) (SENIOR 4) Show that the improper integral

$$\int_2^{+\infty} (x - \sqrt{\lfloor x^2 \rfloor})^2 \, dx$$

is convergent ($\lfloor t \rfloor$ denotes the integer part of, that is, the largest integer not greater than, the real number t).

Source: Adapted from Problem 284 proposed by Gary Walls in College Mathematics Journal; see `http://www.problemcorner.org`

Solution: According to the Mean-Value Theorem of Differentiation, given real numbers a and b with $a<b$ and a function f that is continuous on $[a,b]$ and differentiable on (a,b), we have

$$f(b) - f(a) = f'(\xi)(b-a)$$

with some $\xi \in (a,b)$. Let $x \geq 2$ be such that x^2 is not an integer. Using the Mean-Value Theorem with $a = \lfloor x^2 \rfloor$, $b = x^2$, and $f(t) = \sqrt{t}$, we find that there is a ξ with $\lfloor x^2 \rfloor < \xi < x^2$ such that

$$x - \sqrt{\lfloor x^2 \rfloor} = \sqrt{x^2} - \sqrt{\lfloor x^2 \rfloor} = \frac{1}{2\sqrt{\xi}}(x^2 - \lfloor x^2 \rfloor).$$

Since $\xi > \lfloor x^2 \rfloor > x^2 - 1 \geq (x-1)^2$, we have $\sqrt{\xi} > x-1$; further, clearly $x^2 - \lfloor x^2 \rfloor < 1$ holds; so the right-hand side above is less than $1/(2(x-1))$. Thus

$$x - \sqrt{\lfloor x^2 \rfloor} < \frac{1}{2(x-1)}.$$

We derived this inequality under the assumption that x^2 is not an integer (and $x \geq 2$); but it is obviously true even when x^2 is an integer (since the left-hand side is 0 then). Hence the improper integral in question is dominated by the convergent integral

$$\int_2^{+\infty} \frac{1}{4(x-1)^2} \, dx,$$

showing that the former improper integral is indeed convergent.

9) (SENIOR 5) In the triangle ABC with circumcenter O (i.e., O is the center of the circle going through the vertices of the triangle ABC) we have $AB = AC$, D is the midpoint of the side AB, and E is the centroid of the triangle ACD (the centroid of the triangle is the common intersection of each of three lines connecting a vertex with the midpoint of the opposite side). Prove that OE is perpendicular to CD.

[2] *A fortiori* is new Latin meaning "for a still stronger reason," a phrase frequently used in mathematical writing.

Hint. An algebraic solution using complex numbers works well.

Source: British Mathematical Olympiad, Problem 18. Britain 1983/1. See

`http://www.problemcorner.org`

Solution: In setting up a solution using complex numbers, let the points A, B, C, D, E, O, be represented by the complex numbers a, b, c, d, e, 0, (the last one is the number zero). The assumption that O is the center of the circle going through the points A, B, and C then means that $|a| = |b| = |c|$. There is no harm in assuming that $a = -1$, in which case the equality $AB = AC$ means that B and C are reflections of each other about the real line, i.e., that $c = \bar{a}$ (the bar means complex conjugate). Since $|b| = |c| = |a| = 1$, this means that $bc = 1$. We have

$$d = \frac{a+b}{2} \quad \text{and} \quad e = \frac{a+c+d}{3} = \frac{a+c+\frac{a+b}{2}}{3} = \frac{3a+b+2c}{6}$$

The vector from the point C to the point D is the position vector of the complex number

$$d - c = \frac{a+b}{2} - c = \frac{a+b-2c}{2}$$

(the position vector of a complex number is the vector from the origin to the point represented by the complex number). It is sufficient to prove that the position vectors of $6d$ and $2(c-d)$ are perpendicular, or that the position vectors of $6d \cdot b$ and $2(c-d) \cdot b$ are perpendicular (multiplying by a real number does not change the direction of a vector, and multiplying each vector by the complex number b turns each vector by the same angle). Noting that $a = -1$ and $bc = 1$, we have

$$6ad = b^2 - 3b + 2 = (b-1)(b-2) \quad \text{and} \quad 2(c-3) = b^2 - b - 2 = (b+1)(b-2).$$

In showing that the position vectors of these two complex numbers are perpendicular, we can divide by the common factor $b-2$ (which is not zero, since $|b| = 1$), since this turns both position vectors by the same amounts. That is, we need to show that the position vectors of of the complex numbers $b-1$ and $b+1$ are perpendicular. To show this, we need to verify that the fraction

$$\frac{b-1}{b+1} = \frac{(b-1)(\bar{b}+1)}{(b+1)(\bar{b}+1)}$$

is imaginary. The denominator here is of course real (since it is the product of a complex number and its conjugate). As for the numerator, it equals

$$b\bar{b} + b - \bar{b} - 1 = 1 + b - \bar{b} - 1 = b - \bar{b},$$

and this is clearly imaginary, showing that the fraction itself is also imaginary. This completes the proof.

10) (SENIOR 6) Show that in a convex polyhedron there are always two faces with the same number of sides.

Source: Középiskolai Mathematikai Lapok (The Hungarian Mathematics Journal for High Schools), Problem F. 2484 (1984). See

`http://www.sulinet.hu/cgi-bin/db2www/lm/komal/feladat?id=53884&l=`

In Hungarian. The Web site also provides also has an English translation, see

`http://www.komal.hu/info/bemutatkozas.e.shtml`

Solution: Write E for the number of edges, F for the number of faces, and V for the number of vertices of the polyhedron. Assume that the i face $(1 \le i \le F)$ of the polyhedron has s_i sides. Then

$$2E = \sum_{i=1}^{F} s_i. \tag{1}$$

This is because the right-hand side adds up the number of vertices adjacent to any face; since an edge is adjacent to exactly two faces, this sum gives $2E$. Further, we have

$$3V \le \sum_{i=1}^{F} s_i. \tag{2}$$

This is because the right-hand side also adds up the number of vertices adjacent to any face, and each vertex is adjacent to at least three faces (so the sum counts each vertex at least three times). Finally, if no two faces have the same number of sides then, numbering the faces in a way that the number of sides s_i of the ith face form an increasing sequence, we must have

$$3 \le s_1 < s_2 < \ldots < s_F,$$

and so $s_i \ge 2 + i$ for each i. Hence

$$\sum_{i=1}^{F} s_i \ge \sum_{i=1}^{F} (i+2) = 2F + \frac{F(F+1)}{2} = \frac{F^2 + 5F}{2}. \tag{3}$$

According to Euler's formula we have $V + F - E = 2$, and so, using (1), (2), and (3), we have

$$2 = V + F - E \le \frac{1}{3}\sum_{i=1}^{F} s_i + F - \frac{1}{2}\sum_{i=1}^{F} s_i = F - \frac{1}{6}\sum_{i=1}^{F} s_i \le F - \frac{F^2+5F}{12} = -\frac{F^2 - 7F}{12}.$$

Comparing the extreme members here gives the inequality

$$F^2 - 7F + 24 \le 0.$$

No real number F satisfies this inequality. Hence a polyhedron in which there are no two faces with the same number of sides cannot exist.

11) (SENIOR 7) Let $f(x)$ be a function that is differentiable infinitely many times in $(-\infty, \infty)$. Assume that $f^{(n)}(0) = 0$ and $f^{(n)}(x) \ge 0$ for every integer $n \ge 0$ and every real $x \ge 0$. ($f^{(n)}(x)$ denotes the nth derivative of f. The zeroth derivative $f^{(0)}(x)$ is, of course, $f(x)$ itself.) Show that $f(x) = 0$ for every $x > 0$.

Note. Observe that the first condition $f^{(n)}(0) = 0$ for every integer $n \ge 0$ is not enough the ensure the validity of the conclusion without the second condition. This is shown by the function

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \ne 0, \\ 0 & \text{if } x = 0. \end{cases}$$

For this function, we have $f^{(n)}(0) = 0$ for every integer $n \geq 0$, but the second condition fails: for example, $f'(1) = -2e$ and $f''(1) = 10e$.

Source: This was probably known to S. N. Bernstein, but I have no sources.

Solution: We claim that, for all nonnegative integers n and K, and every positive real x, we have

$$Kf^{(n)}(x) \leq xf^{(n+1)}(x). \tag{1}$$

This implies that $f(x) = 0$ for all $x > 0$, as we are about to show now. Consider this inequality with $n = 1$, in which case it says that

$$Kf(x) \leq xf'(x). \tag{2}$$

Noting that both f and f' are nondecreasing (since their derivatives are nonnegative), and so we must have $f(x) \geq 0$ and $f'(x) \geq 0$ for any $x > 0$. We want to prove that $f(x) = 0$. If this is not the case for a fixed $x > 0$, then let K be a positive integer such that

$$K > \frac{xf'(x)}{f(x)}.$$

For such a K, (2) must fail; this is a contradiction, showing that $f(x) = 0$. Since $x > 0$ is arbitrary here, the conclusion that $f(x) = 0$ for all $x > 0$ follows.

In order to establish (1), we will use induction on K. For $K = 0$, (1) just says that $xf^{(n+1)}(x) \geq 0$; this holds since we assumed that $f^{(n+1)}(x)$ is nonnegative. Assume that (1) holds with a certain nonnegative integer K for all $n \geq 0$ and all $x > 0$. Then we have

$$\begin{aligned} Kf^{(n)}(x) &= \int_0^x Kf^{(n+1)}(t)\,dt \leq \int_0^x tf^{(n+2)}(t)\,dt = xf^{(n+1)}(x) - \int_0^x f^{(n+1)}(t)\,dt \\ &= xf^{(n+1)}(x) - f^{(n)}(x); \end{aligned}$$

here the inequality holds according to (1) with t replacing x and $n + 1$ replacing n; the second equality was obtained via integration by parts. Rearranging this inequality, we obtain

$$(K + 1)f^{(n)}(x) \leq xf^{(n+1)}(x);$$

i.e., (1) holds with $K + 1$ replacing K. This completes the induction step in the proof of (1).