Junior Prize Exam
Spring 2006

1) Given a positive integer n, show that $10n^3 + 3n^2 - n$ is divisible by 6.

2) A function f defined on the interval $[0, 1]$ satisfies $f(0) = f(1)$ and is such that for any x and y with and $0 \le x < y \le 1$ we have

$$|f(y) - f(x)| < y - x.$$

Show that we have

$$|f(x) - f(y)| < \frac{1}{2}$$

whenever $0 \le x < y \le 1$.

3) Given 5 points in a square with side a, show that two of them are within a distance of at most $a/\sqrt{2}$ of each other.

4) Given numbers x_1, x_2, $\dots$ x_n, x_{n+1} such that $x_{n+1} = x_1$, x_i is $+1$ or -1 for each i with $1 \le i \le n + 1$, and

$$\sum_{i=1}^{n} x_i x_{i+1} = 0,$$

show that n is divisible by 4.

5) Show that

$$\sum_{i=1}^{n} \frac{1}{i} = 1 + \frac{1}{2} + \dots + \frac{1}{n}$$

is not an integer for any integer $n \ge 2$.

6) Suppose $f(x)$ and $g(x)$ are nonzero real polynomials satisfying

$$f(x^2 + x + 1) = g(x)f(x).$$

Show that $f(x)$ has even degree.

7) In the complex numbers, the polynomial $x^2 + y^2$ can be factored as $(x + iy)(x - iy)$. Show that the polynomial $x^2 + y^2 + z^2$ cannot be written as a product

$$(ax + by + cz)(Ax + By + Cz),$$

where a, b, c, and A, B, C are complex numbers.

Soon after the exam, solutions will appear on the Web site
http://www.sci.brooklyn.cuny.edu/~mate/prize06/index.html