Senior Prize Exam
Spring 2006

1) Given a positive integer n, show that $10n^3 + 3n^2 - n$ is divisible by 6.

2) A function f defined on the interval $[0,1]$ satisfies $f(0) = f(1)$ and is such that for any x and y with and $0 \le x < y \le 1$ we have

$$|f(y) - f(x)| < y - x.$$

Show that we have

$$|f(x) - f(y)| < \frac{1}{2}$$

whenever $0 \le x < y \le 1$.

3) Given 5 points in a square with side a, show that two of them are within a distance of at most $a/\sqrt{2}$ of each other.

4) Show that the improper integral

$$\int_2^{+\infty} (x - \sqrt{\lfloor x^2 \rfloor})^2 \, dx$$

is convergent ($\lfloor t \rfloor$ denotes the integer part of, that is, the largest integer not greater than, the real number t).

5) In the triangle ABC with circumcenter O (i.e., O is the center of the circle going through the vertices of the triangle ABC) we have $AB = AC$, D is the midpoint of the side AB, and E is the centroid of the triangle ACD (the centroid of the triangle is the common intersection of each of three lines connecting a vertex with the midpoint of the opposite side). Prove that OE is perpendicular to CD.

Hint. An algebraic solution using complex numbers works well.

6) Show that in a convex polyhedron there are always two faces with the same number of sides.

7) Let $f(x)$ be a function that is differentiable infinitely many times in $(-\infty, \infty)$. Assume that $f^{(n)}(0) = 0$ and $f^{(n)}(x) \ge 0$ for every integer $n \ge 0$ and every real $x \ge 0$. ($f^{(n)}(x)$ denotes the nth derivative of f. The zeroth derivative $f^{(0)}(x)$ is, of course, $f(x)$ itself.) Show that $f(x) = 0$ for every $x > 0$.

Note. Observe that the first condition $f^{(n)}(0) = 0$ for every integer $n \ge 0$ is not enough the ensure the validity of the conclusion without the second condition. This is shown by the function

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \ne 0, \\ 0 & \text{if } x = 0. \end{cases}$$

For this function, we have $f^{(n)}(0) = 0$ for every integer $n \ge 0$, but the second condition fails: for example, $f'(1) = -2e$ and $f''(1) = 10e$.

Soon after the exam, solutions will appear on the Web site
`http://www.sci.brooklyn.cuny.edu/~mate/prize06/index.html`