All Problems on Prize Exam
Spring 2007
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The source for each problem is listed below when available; but even when the source is given, the formulation of the problem may have been changed. Solutions for the problems presented here were obtained without consulting sources for these solutions even when available, and additional information on how to solve these problems might be obtained by consulting the original sources. There was some overlap between the problems on the Junior and Senior prize exams; the problems common to both exams are listed only once.

1) (JUNIOR 1 and SENIOR 1) Let $n > 1$ be a positive integer such that 2^n and 5^n start with the same digit in their decimal expansion. Show that this starting digit must be 3. (The numbers are written without leading zeros.)

Source: Problem 122,
`http://www.qbyte.org/puzzles/`

Solution: If d is the starting digit, then for some k and l we have

$$10^k d < 2^n < (d+1)10^k,$$
$$10^l\, d < 5^n < (d+1)10^l$$

(we have strict inequalities on the left since neither 2^n nor 5^n is a multiple of a power of 10). Multiplying these inequalities together, we obtain

$$10^{k+l} d^2 < 10^n < (d+1)^2 10^{k+l}.$$

As $1 \le d \le 9$, this is possible only if $n = k + l + 1$ and $d^2 < 10 < (d+1)^2$. Hence $d = 3$; i.e., the common starting digit must be 3.

Note that $2^5 = 32$ and $5^5 = 3125$, showing that numbers satisfying the requirement do indeed exist.

2) (JUNIOR 2 and SENIOR 2) In a circle with center O, two radii OA and OB are given. Describe how to draw a chord (using only a ruler and a compass) that is divided into three equal parts by the radii OA and OB.

Source: Középiskolai Matematikai Lapok (The Hungarian Mathematics Journal for High Schools), Problem 134 (1895/4). See
`http://www.sulinet.hu/komal/`

Solution: Find points C and D on the line AB such that the distances AC, AB, and BD are equal, and such that the points A, B, C, and D are distinct. Let C' and D' be the intersections of the lines OC and OD with the circle, respectively. Then the chord $C'D'$ is divided into three equal parts by the radii OA and OB, as this can easily be proved by using similar triangles.

3) (JUNIOR 3 and SENIOR 3) Let $P(x)$ be a polynomial with real integer coefficients. Assume that $P(0)$ and $P(1)$ are both odd numbers. Show that the equation $P(x) = 0$ cannot have a root that is a real integer.

Source: American Mathematical Monthly, Problem AL-327 by V. M. Spunar

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http://www.problemcorner.org

Solution: Assume that the (real) integer n is a root of the equation $P(x) = 0$. Then $P(x) = (x-n)Q(x)$ for some polynomial $Q(x)$ with real integers as its coefficients (to see that the coefficients of $Q(x)$ are integers, note that $Q(x)$ can be obtained by dividing $x - n$ into $P(x)$, and at each step of this division, we obtain an integer coefficient). Now, $P(0) = (-n)Q(0)$ and $P(1) = (1 - n)Q(1)$. Since either $-n$ of $1 - n$ is an even number, at least one of these numbers must be even (since $Q(0)$ and $Q(1)$ are integers).

4) (JUNIOR 4) Let A and B be positive integers, and assume that the arithmetic progression $\{An + B : n = 0, 1, \dots\}$ contains at least one square of an integer. If M^2 ($M > 0$) is the least such square, prove that $M < A + \sqrt{B}$.

Source: Crux Mathematicorum 1166, by Kenneth S. Williams

http://www.problemcorner.org

Solution: Assume, on the contrary, that $M \geq A + \sqrt{B}$. Then

$$(M - A)^2 = M^2 - 2MA + A^2 = M^2 + A(A - 2M).$$

So, if $M^2 = An_1 + B$, then $(M - A)^2 = An_2 + B$ with $n_2 = n_1 + A - 2M$. To complete the proof, we only need to show that $n_2 \geq 0$ (because this would contradict the assumption that M^2 is the smallest integer of form $An + B$ with $n \geq 0$).

We have $M - A \geq \sqrt{B}$ by our assumption, so

$$An_2 + B = (M - A)^2 \geq B;$$

hence $n_2 \geq 0$, as we wanted to show.

5) (JUNIOR 5) Let f be a real-valued function on the real line such that $f(x) \leq x$ and $f(x+y) \leq f(x) + f(y)$ for all real numbers x and y. Show that $f(x) = x$ for all x.

Source: Problem 2 at

http://www.geocities.com/CapeCanaveral/Lab/4661/Frame_Calculus.html

Solution: We have

$$f(0) = f(0 + 0) \leq f(0) + f(0),$$

and this implies $f(0) \geq 0$. Since we also have $f(0) \leq 0$ according to the assumptions, it follows that $f(0) = 0$. Hence

$$0 = f(0) = f(x + (-x)) \leq f(x) + f(-x) \leq x + (-x) = 0;$$

this implies that $f(-x) = -f(x)$. Again, by one of the assumptions, we have

$$-f(x) = f(-x) \leq -x;$$

this means that $f(x) \geq x$. As we also have $f(x) \leq x$ by our assumptions, $f(x) = x$ follows.

6) (JUNIOR 6) Show that

$$\cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} = -\frac{1}{2}.$$

Source: Problem 5 at

http://www.geocities.com/CapeCanaveral/Lab/4661/Frame_Trigonometry.html

Solution: With

$$\zeta = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5},$$

where $i = \sqrt{-1}$, we have

$$\zeta^5 = \cos 2\pi + i\sin 2\pi = 1$$

according to the de Moivre formula, i.e.,

$$\zeta^5 - 1 = (\zeta - 1)(1 + \zeta + \zeta^2 + \zeta^3 + \zeta^4) = 0.$$

Since the first factor on the right-hand side is not zero, the second factor must be zero; that is,

$$(\zeta + \zeta^4) + (\zeta^2 + \zeta^3) = -1.$$

As $\zeta \cdot \zeta^4 = \zeta^5$, the numbers ζ and ζ^4 are complex conjugates. Similarly, ζ^2 and ζ^3 are complex conjugates. Hence, writing $\Re z$ for the real part of the complex number z, we have

$$\zeta + \zeta^4 = 2\Re\zeta = 2\cos\frac{2\pi}{5}.$$

Similarly,

$$\zeta^2 + \zeta^3 = 2\Re\zeta^2 = 2\cos\frac{4\pi}{5}.$$

Thus, the above equation can be written as

$$2\cos\frac{2\pi}{5} + 2\cos\frac{4\pi}{5} = -1,$$

which is equivalent to the equality to be proved.

Note. There are various ways to eliminate complex numbers from the above proof, appealing to geometric insights about the regular pentagon, for example. But this would not change the basic ideas behind the proof.

7) (JUNIOR 7) Let $r_1, r_2, \ldots, r_n$ be n positive integers ($n > 1$) with $r_1 \le r_2 \le r_3 \le \ldots \le r_n$ be such that

$$\frac{1}{r_1} + \frac{1}{r_2} + \ldots + \frac{1}{r_n} = 1.$$

Show that $r_i \le n^{(2^{i-1})}$ for each i with $1 \le i \le n$.

Source: Középiskolai Matematikai Lapok (The Hungarian Mathematics Journal for High Schools), Problem N. 65 (1995/4). See

`http://www.sulinet.hu/komal/`

Solution: Let k with $1 \le k \le n$ be the least integer for which $r_k > n^{2^{k-1}}$ (note that $n^{2^k} \stackrel{def}{=} n^{(2^k)}$, i.e., the use of the parentheses in the exponent is not required). Then

$$\frac{1}{r_k} + \frac{1}{r_{k+1}} + \ldots + \frac{1}{r_n} \le (n-k+1)\cdot\frac{1}{r_k} < n \cdot n^{-2^{k-1}} = n^{-2^{k-1}+1}$$

This shows that we must have $k > 1$. Indeed, if we had $k = 1$ then the left-hand side would be 1, and the right-hand side is also 1; in view of the second inequality being a strict inequality, this is a contradiction.

On the other hand, given that $k > 1$,[1] we have

$$r_1 r_2 \dots r_{k-1} \le n^{2^0+2^1+\dots+2^{k-2}} = n^{2^{k-1}-1}.$$

Now,

$$1 - \frac{1}{r_1} - \frac{1}{r_2} - \dots - \frac{1}{r_{k-1}} = \frac{a}{r_1 r_2 \cdot r_{k-1}} \ge n^{-2^{k-1}+1};$$

indeed, the equality holds with some positive integer a if one carries out the operations on the left-hand side by taking $r_1 r_2 \dots r_{k-1}$ as the common denominator. The inequality holds in view of the preceding inequality if one notes that $a \ge 1$. But then the equality

$$1 - \frac{1}{r_1} - \frac{1}{r_2} - \dots - \frac{1}{r_{k-1}} = \frac{1}{r_k} + \frac{1}{r_{k+1}} + \dots + \frac{1}{r_n}$$

cannot hold. Indeed, the left-hand side is greater than the right-hand side, in view of the first and third displayed inequalities above (note the strict inequality in the first displayed inequality).

8) (SENIOR 4) For a positive integer n, put

$$a_n = \int_0^n e^{t^2/n}\,dt.$$

Show that $\lim_{n\to\infty} a_{n+1}/a_n = e$.

Source: Elemente der Mathematatik 892, by U. Abel,

`http://www.problemcorner.org`

Solution: Using the substitution $u = t/\sqrt{x}$, we have

$$a(x) \stackrel{def}{=} \int_0^x e^{t^2/x}\,dt = \frac{1}{\sqrt{x}} \int_0^{\sqrt{x}} e^{u^2}\,du.$$

Hence we have

$$\lim_{x\to\infty} \frac{a(x+1)}{a(x)} = \lim_{x\to\infty} \frac{\sqrt{x}}{\sqrt{x+1}} \cdot \lim_{x\to\infty} \frac{\int_0^{\sqrt{x+1}} e^{u^2}\,du}{\int_0^{\sqrt{x}} e^{u^2}\,du}.$$

The first limit on the right-hand side is 1, and for the second limit we can use l'Hospital's rule, since the limit of both the numerator and the denominator is infinite. Hence

$$\lim_{x\to\infty} \frac{a(x+1)}{a(x)} = \lim_{x\to\infty} \frac{\frac{1}{2\sqrt{x+1}} e^{x+1}}{\frac{1}{2\sqrt{x}} e^x} = e,$$

as we wanted to show.

9) (SENIOR 5) Let x, y, z be real numbers with $x < y < z$, and assume the function is continuous in the interval $[x, z]$ and $f''(u) > 0$ for all $u \in (x, z)$. Show that

$$f(x)(y-z) + f(y)(z-x) + f(z)(x-y) < 0.$$

[1] The following argument will also work for $k = 1$ if we take the empty product to be 1 and the empty sum to be 0. So there was no need to point out above that $k > 1$.

Source: Problem 2213, Crux Mathematicorum, Vol. 23 (1997), proposed by Victor Oxman; see `http://journals.cms.math.ca/cgi-bin/vault/public/view/CRUXv23n1/body/HTML/48?template=CRUX`
or

`http://journals.cms.math.ca/CRUX/`

Solution: By the Mean-Value Theorem, there are $\xi \in (x, y)$ and $\eta \in (y, z)$ such that

$$f'(\xi) = \frac{f(x) - f(y)}{x - y} \qquad \text{and} \qquad f'(\eta) = \frac{f(y) - f(z)}{y - z}.$$

f' is increasing in (x, z), since $f'' > 0$ in this interval. As $\xi < \eta$, we therefore have $f'(\xi) < f'(\eta)$. In view of the above equations, this can be written as

$$\frac{f(x) - f(y)}{x - y} < \frac{f(y) - f(z)}{y - z}.$$

That is, multiplying both sides by $(x - y)(y - z)$ (which is a positive number, being the product of two negative numbers), we obtain

$$f(x)(y - z) - f(y)(y - z) < f(y)(x - y) - f(z)(x - y).$$

Rearranging this, and canceling the term $-f(y)y$, we obtain the inequality to be proved.

10) (SENIOR 6) Let P_1, P_2, ..., P_n be n points on a circle of radius 1. Prove that there exists a point Q on the circle such that

$$\sum_{i=1}^{n} \overline{QP_i} > \frac{4n}{\pi}.$$

Source: The College Mathematics Journal, Problem 372, by Eugene Levine,

`http://www.problemcorner.org`

Solution: Take the circle to be the unit circle of the coordinate system. Let θ_i be the polar angle of the point P_i, and let θ be the polar angle of the point Q. Then $\overline{QP_i} = 2\left|\sin \frac{\theta-\theta_i}{2}\right|$. As Q moves around the circumference of the unit circle, the average of this distance is

$$\frac{1}{2\pi}\int_0^{2\pi} 2\left|\sin\frac{\theta-\theta_i}{2}\right| d\theta = \frac{1}{\pi}\int_{-\theta_i}^{2\pi-\theta_i}\left|\sin\frac{t}{2}\right| dt = \frac{1}{\pi}\int_0^{2\pi}\left|\sin\frac{t}{2}\right| dt = \frac{1}{\pi}\int_0^{2\pi}\sin\frac{t}{2}\, dt = \frac{4}{\pi};$$

here, the second equality holds since $\left|\sin\frac{t}{2}\right|$ is periodic, with period 2π, and after the third equality we dropped the absolute value sign since $\sin\frac{t}{2}$ is nonnegative on $[0, 2\pi]$. Hence the average of $\sum_{i=1}^{n} \overline{QP_i}$ as Q moves around the circumference of the unit circle is $4n/\pi$. Hence, for some Q we must have

$$\sum_{i=1}^{n} \overline{QP_i} \geq \frac{4n}{\pi}.$$

In fact, there must be a Q for which strict inequality holds here, unless for every Q we have equality.[2] Since it is not possible to have equality for all Q, there must be a Q for which strict inequality holds.

[2] Indeed, if we have $\leq$ for every Q, this would mean that the of the right-hand side minus the left-hand side of this inequality is always nonnegative, while its integral as Q circumnavigates the unit circle is 0. For a continuous function, this is possible only if it is identically zero.

This is because

$$\sum_{i=1}^{n} 2\left|\sin\frac{\theta-\theta_i}{2}\right|$$

cannot be constant, since its derivative is not always zero. To see this, may assume that $0 \le \theta_i < 2\pi$ for each i. In fact, we may even assume $0 < \theta_i < 2\pi$ (indeed, if we had $\theta_i = 0$ for some i, then we can replace each θ_i with $\theta_i + \theta_0$ for a small $\theta_0 > 0$ such that $\theta_i + \theta_0 < 2\pi$ for each i; this amounts to turning the coordinate system by an angle $-\theta_0$). If we choose θ with $\theta_i < \theta < 2\pi$ for each i, then we can drop the absolute value from the the above expression:

$$\sum_{i=1}^{n} 2\sin\frac{\theta-\theta_i}{2}$$

This derivative is

$$\sum_{i=1}^{n} \cos\frac{\theta-\theta_i}{2} = \cos\frac{\theta}{2}\cdot\sum_{i=1}^{n}\cos\frac{\theta_i}{2} + \sin\frac{\theta}{2}\cdot\sum_{i=1}^{n}\sin\frac{\theta_i}{2}.$$

Every term in the sum multiplying $\sin(\theta/2)$ is positive, so this expression cannot be identically zero on any interval. Indeed, $A\cos t + B\sin t$ can be identically zero on an interval only if $A = B = 0$. This is because we can write

$$A\cos t + B\sin t = C\cos(t - t_0)$$

with $C = \sqrt{A^2 + B^2}$, and t_0 so determined that $\cos t_0 = A/C$ and $\sin t_0 = B/C$. Since this expression is the derivative of the sum in question on a nonempty interval, this shows that the sum in question cannot be constant.

11) (SENIOR 7) A straight rod is cut at random into three pieces. What is the probability that one of these pieces is longer than half the length of the original rod. (The way to imagine the random cutting is as follows: pick two points at random along the length of the rod, and then cut the rod at these places – rather than cutting the rod at one place and then at another place, since in this case one would have to first decide which of the two pieces to cut.)

Source: Spectrum, Problem 14.1, at

`http://www.problemcorner.org`

Solution: We may assume that the length of the rod is 1. The cut points may be represented by a pair of numbers (x, y), the distances of the cuts being measured from the left end point of the rod; then $0 < x, y < 1$. The total area represented by the pair of the points (x, y) is 1, the area of the unit rectangle. Probabilities are represented by areas corresponding to a favorable outcome.

After the cutting, the three resulting pieces are the intervals $[0, \min(x, y)]$, $[\min(x, y), \max(x, y)]$ and $[\max[x, y], 1]$. In the present case, the outcome is favorable if

(1) $x, y < 1/2$, when the third piece is longer that $1/2$;
(2) $x, y > 1/2$, when the first piece is longer than $1/2$;
(3) $y < x - 1/2$, when the second piece is longer than $1/2$;
(4) $y > x + 1/2$, when again the second piece is longer than $1/2$.

Any two of these regions are disjoint. The regions represented by the first and fourth cases are squares, and those by the second and third cases are triangles. The areas of the squares are $1/4$ each, and the areas of the triangles are $1/8$ each. The total area represented by these is $2\cdot\frac{1}{4} + 2\cdot\frac{1}{8} = 3/4$. That is, the probability of a favorable outcome is $3/4$.

Note. If one first cuts the rod into two pieces, and then chooses, with equal probability, one of the two pieces to do a further cut, then the result will be different. Namely, if one cuts the shorter piece, the outcome will be favorable in that there will be a piece of length greater than $1/2$. This happens with probability $1/2$. If one cuts the longer piece, and the length of this piece is $x\,(\geq 1/2)$, then the (conditional) probability of a favorable outcome is

$$\frac{(x-1/2)+(x-1/2)}{x} = 2 - \frac{1}{x};$$

the numerator in the left-a-hand side corresponds to selecting the second cut point close to each of the endpoints, so that the other part has length $> 1/2$. The probability on condition that the longer piece is cut is then the integral of these conditional probabilities:

$$\int_{1/2}^{1} \left(2 - \frac{1}{x}\right)\,dx = \frac{1}{2} - \left(\log 1 - \log\frac{1}{2}\right) = \frac{1}{2} - \log 2;$$

here log denotes the natural logarithm.[3] This probability must then be multiplied by $1/2$, the probability that the longer piece is cut, and then added to to this the probability $1/2$ obtained above, when the shorter piece was cut. Thus, the total probability is

$$\frac{1}{2} + \frac{1}{2}\left(\frac{1}{2} - \log 2\right) = \frac{3}{4} - \frac{1}{2}\log 2.$$

This shows that this way of cutting the rod is very different from the random cutting described in the problem.

[3]Mathematicians almost never use $\ln x$ to denote the natural logarithm of x; the common custom is to use $\log x$ for this. $\ln x$ is commonly used in some fields of science where mathematics is applied. $\log x$ is never used in mathematics to denote logarithm of base 10 of x; only calculators use $\log x$ to denote logarithm of base 10 of x. Base 10 logarithm has practically no use in mathematics, while it is used in some sciences where logarithmic scale is used for measurements – such as for measuring earthquakes, or pH in chemistry. Base 10 logarithm was convenient for calculations when logarithm tables were used to speed up multiplication, but with calculators and computers the use of logarithm tables has become obsolete. The notation $\lg x$ was often used to denote logarithm of base 10; the advantage of this notation is that it in not in conflict with the use of $\log x$.