Junior Prize Exam
Spring 2007

1) Let $n > 1$ be a positive integer such that 2^n and 5^n start with the same digit in their decimal expansion. Show that this starting digit must be 3. (The numbers are written without leading zeros.)

2) In a circle with center O, two radii OA and OB are given. Describe how to draw a chord (using only a ruler and a compass) that is divided into three equal parts by the radii OA and OB.

3) Let $P(x)$ be a polynomial with real integer coefficients. Assume that $P(0)$ and $P(1)$ are both odd numbers. Show that the equation $P(x) = 0$ cannot have a root that is a real integer.

4) Let A and B be positive integers, and assume that the arithmetic progression $\{An + B : n = 0, 1, \dots\}$ contains at least one square of an integer. If M^2 ($M > 0$) is the least such square, prove that $M < A + \sqrt{B}$.

5) Let f be a real-valued function on the real line such that $f(x) \le x$ and $f(x+y) \le f(x) + f(y)$ for all real numbers x and y. Show that $f(x) = x$ for all x.

6) Show that

$$\cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} = -\frac{1}{2}.$$

7) Let r_1, r_2, ..., r_n be n positive integers ($n > 1$) with $r_1 \le r_2 \le r_3 \le \ldots \le r_n$ be such that

$$\frac{1}{r_1} + \frac{1}{r_2} + \ldots + \frac{1}{r_n} = 1.$$

Show that $r_i \le n^{(2^{i-1})}$ for each i with $1 \le i \le n$.

Soon after the exam, solutions will appear on the Web site
http://www.sci.brooklyn.cuny.edu/~mate/prize07/index.html

All computer processing for this manuscript was done under Fedora Core Linux. $\mathcal{A}\mathcal{M}\mathcal{S}$-TeX was used for typesetting.