

SENIOR PRIZE EXAM
SPRING 2007

1) Let $n > 1$ be a positive integer such that 2^n and 5^n start with the same digit in their decimal expansion. Show that this starting digit must be 3. (The numbers are written without leading zeros.)

2) In a circle with center O , two radii OA and OB are given. Describe how to draw a chord (using only a ruler and a compass) that is divided into three equal parts by the radii OA and OB .

3) Let $P(x)$ be a polynomial with real integer coefficients. Assume that $P(0)$ and $P(1)$ are both odd numbers. Show that the equation $P(x) = 0$ cannot have a root that is a real integer.

4) For a positive integer n , put

$$a_n = \int_0^n e^{t^2/n} dt.$$

Show that $\lim_{n \rightarrow \infty} a_{n+1}/a_n = e$.

5) Let x, y, z be real numbers with $x < y < z$, and assume the function is continuous in the interval $[x, z]$ and $f''(u) > 0$ for all $u \in (x, z)$. Show that

$$f(x)(y - z) + f(y)(z - x) + f(z)(x - y) < 0.$$

6) Let P_1, P_2, \dots, P_n be n points on a circle of radius 1. Prove that there exists a point Q on the circle such that

$$\sum_{i=1}^n \overline{QP_i} > \frac{4n}{\pi}.$$

7) A straight rod is cut at random into three pieces. What is the probability that one of these pieces is longer than half the length of the original rod. (The way to imagine the random cutting is as follows: pick two points at random along the length of the rod, and then cut the rod at these places – rather than cutting the rod at one place and then at another place, since in this case one would have to first decide which of the two pieces to cut.)

SOON AFTER THE EXAM, SOLUTIONS WILL APPEAR ON THE WEB SITE

<http://www.sci.brooklyn.cuny.edu/~mate/prize07/index.html>