All Problems on Prize Exam
Spring 2008
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The source for each problem is listed below when available; but even when the source is given, the formulation of the problem may have been changed. Solutions for the problems presented here were obtained without consulting sources for these solutions even when available, and additional information on how to solve these problems might be obtained by consulting the original sources. There was some overlap between the problems on the Junior and Senior prize exams; the problems common to both exams are listed only once.

1) (JUNIOR 1 and SENIOR 1) Prove that the sum of the squares of any five consecutive integers is divisible by 5.

Source: Problem 5.1.4, Journal: Function, Publisher: Monash University, Proposal 5(1981/1)28 by Garnet J. Greenbury, Solution 5(1981/4)31 by Wen Ai Soong. See

`http://www.problemcorner.org`

Solution: We may assume that the five numbers x_0, x_1, x_2, x_3, x_4 satisfy $x_k \equiv k \mod 5$ for $k = 0, 1, 2, 3, 4$.

Note that we are not saying that the numbers are listed in order, that is we are not saying that $x_k = x_0 + k$. For example, it may be that $x_0 = 15$, $x_1 = 16$, $x_2 = 12$, $x_3 = 13$, $x_4 = 14$. Nevertheless, the five numbers will each give a different remainder when divided by 5. In other words, the numbers may be listed as a cyclic permutation of their natural order.

We can square and then add the congruences $x_k \equiv k \mod 5$ to obtain

$$x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 \equiv 0^2 + 1^2 + 2^2 + 3^2 + 4^2 \mod 5.$$

The right-hand side here is 30, which is divisible by 5. So the left-hand side is also divisible by 5, as we wanted to show.

Second solution. Let the numbers be $n-2$, $n-1$, n, $n+1$, and $n+2$, where n is an integer. Then

$$\begin{aligned}&(n-2)^2 + (n-1)^2 + n^2 + (n+1)^2 + (n+2)^2 \\ &\qquad = (n^2-4n+4) + (n^2-2n+1) + n^2 + (n^2+2n+1) + (n^2+4n+4) + = 5n^2 + 10,\end{aligned}$$

and this is clearly divisible by 5.

2) (JUNIOR 2 and SENIOR 2) Let f be a function defined for all real x, and suppose that

$$|f(x) - f(y)| \le (x-y)^2$$

for all real x and y. Prove that f is constant.

Source: Walter Rudin, *Principles of Mathematical Analysis*, Third Edition, McGraw-Hill, New York, 1976, Exercise 1 at the end of Chapter 5, p. 114.

Solution: For arbitrary real numbers x and y with $x \ne y$ we have

$$\left|\frac{f(y)-f(x)}{y-x}\right| \le \left|\frac{(y-x)^2}{y-x}\right| = |y-x|;$$

All computer processing for this manuscript was done under Fedora Linux. $\mathcal{A}\mathcal{M}\mathcal{S}$-TeX was used for typesetting.

hence

$$-|y-x| \le \frac{f(y)-f(x)}{y-x} \le |y-z|.$$

If we keep x fixed and make y tend to x, the limits of the expressions on the extreme left and the extreme right are both zero; hence the limit of the expression in the middle must exist and must equal zero. Thus,

$$f'(x) = \lim_{y \to x} \frac{f(y)-f(x)}{y-x} = 0.$$

This is true for every real x. Thus, $f'(x) = 0$ for all x; consequently, f is constant.

3) (JUNIOR 3 and SENIOR 3) Prove that

$$(2^{2^0}+1)(2^{2^1}+1)(2^{2^2}+1)(2^{2^3}+1)\dots(2^{2^n}+1) = 2^{2^{n+1}}-1$$

holds for any integer $n \ge 0$.

Source: Based on Problem E152, American Mathematical Monthly 42 (1935), p. 246. See `http://www.problemcorner.org`

Solution: We will use induction on n. For $n = 0$, the equation says that $2^1 + 1 = 2^2 - 1$, i.e., that $2 + 1 = 4 - 1$, and this is clearly true. Let $n > 0$, and assume that we already know that

$$\prod_{i=0}^{n-1}(2^{2^i}+1) = 2^{2^n}-1.$$

Then

$$\prod_{i=0}^{n}(2^{2^i}+1) = \left(\prod_{i=0}^{n-1}(2^{2^i}+1)\right)(2^{2^n}+1) = (2^{2^n}-1)(2^{2^n}+1) = \left(2^{2^n}\right)^2 - 1 = 2^{2\cdot 2^n}-1 = 2^{2^{n+1}}-1;$$

this completes the proof of the equation by induction. Note that in the second member of the last displayed equation, the parentheses surrounding the product symbol $\prod$ are necessary to limit the scope of the product.

Note. The disadvantage of the above derivation is that it makes use of the fact that the right-hand side of the equation is given.[1] Using some numerical experimentation, one can guess the the right-hand side without too much difficulty. However, there is a direct derivation of the right-hand side using the *generalized distributive rule.* Since the generalized distributive rule is worth knowing about, we will give the details.

To explain what the generalized distributive rule says, let (a_{ij}) be an $m \times n$ matrix. Then

$$\prod_{i=1}^{m}\sum_{j=1}^{n} a_{ij} = \sum_{\sigma}\prod_{i=1}^{m} a_{i\sigma(i)},$$

where σ runs over all mappings of the set $M_m = \{1, 2, \dots, m\}$ into the set $M_n = \{1, 2, \dots, n\}$. The left-hand side represent a product of sums. The right-hand side multiplies out this product by

[1]The way the problem was originally stated in the American Mathematical Monthly, loc. cit. (Latin for *loco citato*, meaning "in the place cited"), the right-hand side was not given.

taking one term out of each of these sums, and adding up all the products that can be so formed. The equality of these two sides is obtained by the distributivity of multiplication over addition. For example, for $n = 2$, the above equation says that

$$(a_{11} + a_{12})(a_{21} + a_{22}) = a_{1\sigma_1(1)}a_{2\sigma_1(2)} + a_{1\sigma_2(1)}a_{2\sigma_2(2)} + a_{1\sigma_3(1)}a_{2\sigma_3(2)} + a_{1\sigma_4(1)}a_{2\sigma_4(2)},$$

where $\sigma_1(1) = 1$, $\sigma_1(2) = 1$, $\sigma_2(1) = 1$, $\sigma_2(2) = 2$, $\sigma_3(1) = 2$, $\sigma_3(2) = 1$, $\sigma_4(1) = 2$, $\sigma_4(2) = 2$.

To apply this rule, let $a_{ij} = 1$ for $0 \le i \le n$ and $j = 0$, and let $a_{ij} = 2^{2^i}$ for $0 \le i \le n$ and $j = 1$. Note that we can also write this as

$$a_{ij} = 2^{j\cdot 2^i} \qquad (0 \le i \le n,\ j = 0, 1).$$

Let σ run over all mappings from the set $\{0, 1, 2, \ldots, n\}$ into set $\{0, 1\}$. Note that we have

$$(2^{2^0} + 1)(2^{2^1} + 1)(2^{2^2} + 1)(2^{2^3} + 1)\ldots(2^{2^n} + 1) = \prod_{i=0}^{n}(2^{2^i} + 1)$$
$$= \prod_{i=0}^{n}\sum_{j=0}^{1} a_{ij} = \sum_{\sigma}\prod_{i=0}^{n} a_{i\sigma(i)} = \sum_{\sigma}\prod_{i=0}^{n} 2^{\sigma(i)\cdot 2^i} = \sum_{\sigma} 2^{\sum_{i=0}^{n}\sigma(i)\cdot 2^i};$$

here the third equality is an application of the distributive rule, and the fourth equality uses the displayed formula for a_{ij}. In the exponent on the right-hand side, the sum $\sum_{i=0}^{n}\sigma(i)\cdot 2^i$ describes the n-digit number written in binary notation whose digit with place value 2^i is $\sigma(i)$. Since each integer N with $0 \le N \le \sum_{i=0}^{n} 2^i = 2^{n+1} - 1$ has exactly one such binary representation (the equation here is based on the sum formula for the geometric progression), the right-hand side of the last displayed formula equals

$$\sum_{N=0}^{2^{n+1}-1} 2^N = 2^{2^{n+1}} - 1,$$

where we again used the sum formula for the geometric progression. The equation claimed in the problem has thus been established.

4) (JUNIOR 4) Find all positive integers x and y such that $x^3 - y^3 = 91$.

Source: Dániel Arany Mathematical Competition, 1996/97, Hungary, Second round, Problem 3 in Category I and Problem 1 in Category II. See
`http://matek.fazekas.hu/portal/feladatbank/egyeb/AranyDani/ad97fel/ad97felm.html`

Solution: We have $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$, so both $x - y$ and $x^2 + xy + y^2$ must be divisors of $91 = 7 \cdot 13$, where 7 and 13 are prime numbers. Now, clearly, $x - y < x^2 + xy + y^2$. So we could try $x - y = 7$, but then we would have to have $x^2 + xy + y^2 = 13$. This is, however, not possible, since $x = y + 7 \ge 8$, and $8^2 > 13$. Hence, the only possibility that remains is $x - y = 1$ and $x^2 + xy + y^2 = 91$. Then $x = y + 1$, so the latter equation gives

$$(y^2 + 1)^2 + (y + 1)y + y^2 = 91,$$

that is,

$$y^2 + y - 30 = 0.$$

This gives $y = -6$ and $y = 5$. Taking only the positive solution for y, this gives the solution $x = 6$ and $y = 5$ for the above equation.

5) (JUNIOR 5) Let m and n be positive integers such that $\sqrt{m} + \sqrt{n}$ is an integer. Prove that both $\sqrt{m}$ and $\sqrt{n}$ are integers.

Source: No outside source

Solution: Let k be an integer such that $\sqrt{m} + \sqrt{n} = k$. Then

$$k - \sqrt{n} = \sqrt{m}.$$

Squaring both sides, it follows that

$$k^2 - 2k\sqrt{n} + n = m.$$

Rearranging this, we obtain that

$$\sqrt{n} = \frac{m - k^2 - n}{2k}.$$

That is, $\sqrt{n}$ is rational. It therefore follows that $\sqrt{n}$ is actually an integer.

To see this latter point, note that $\sqrt{n}$ is a solution of the equation

$$x^2 - n = 0.$$

If x is rational, then the Rational Roots Test says that x can be written as a fraction p/q where p and q are integers, p is a divisor of n, and q is a divisor of 1. Thus $q = \pm 1$, and so $\sqrt{n} = x = \pm p$ is indeed an integer.

It now follows that $\sqrt{m} = k - \sqrt{n}$ ks also an an integer.

6) (JUNIOR 6) How many ways can you pick two subsets A and B of the seven-element set $C = \{1, 2, 3, 4, 5, 6, 7\}$ such that the set $A \cap B$ is nonempty. **Note:** We want to count the ordered pairs (A, B); that is, we consider the choice $(A, B) = (\{3, 5, 7\}, \{5, 6\})$ different from the choice $(A, B) = (\{5, 6\}, \{3, 5, 7\})$.

Source: No outside source.

Solution: First, we want to count the number of choices (A, B) such that $A \cap B$ is empty. Given such a choice for A and B, define the function $f : C \to \{0, 1, 2\}$ such that, for i with $1 \le i \le 7$, we put $f(i) = 0$ if $i \notin A$ and $i \notin B$, $f(i) = 1$ if $i \in A$ and $i \notin B$, $f(i) = 2$ if $i \notin A$ and $i \in B$. It is clear that the function f is uniquely determined by the choice of A and B, and, conversely, the function f uniquely determines the choice of A and B. Hence, the number of functions f is equal to the number of pairs (A, B) satisfying $A \cap B = \emptyset$. The number of choices for f is $3^7 = 2187$.

The total number ways to pick a subset of C is $2^7 = 128$; so the number of ways to pick a pair of subsets (A, B) of C is $2^7 \cdot 2^7 = 2^{14} = 16384$. Subtracting from this number the number of ways we can choose (A, B) such that $A \cap B = \emptyset$, we obtain $2^{14} - 3^7 = 14197$, the number of ways (A, B) can be chosen such that $A \cap B \neq \emptyset$.

7) (JUNIOR 7) Let A_1, A_2, A_3, A_4 be four distinct points in the plane and consider the set $X = \{A_1, A_2, A_3, A_4\}$. Prove that there exists a subset Y of X with the property that there is no closed disk K such that $K \cap X = Y$. (A closed disk is the set of points inside and on the circumference of a circle.)

Source: Problem 2 in the 18ht Austrian–Polish Mathematics Competition, Austria, June 28-30, 1995, http://cage.rug.ac.be/~hvernaev/Olympiade/APMC95.dvi

Solution: We may assume that no three of the four points lie on the same line. For example, if A_2 were to lie in the line segment $\overline{A_1A_3}$, then any closed disk containing the points A_1 and A_3 would have to contain also the point A_2, and so we could take $Y = \{A_1, A_3\}$ to witness the assertion in the problem (i.e., then Y would be such that there is no closed disk K for which $K \cap X = Y$).

If one point, say A_4, is inside the triangle determined by the other three points. then any disk containing the points A_1, A_2, and A_3 must also contain A_4, showing that we can take $Y = \{A_1, A_2, A_3\}$ to witness the assertion of the problem.

Hence, we may assume that the points A_1, A_2, A_3, A_4 are labeled in such a way that they are the vertices of a convex quadrilateral listed in a cyclic order (i.e., the walk A_1, A_2, A_3, A_4 takes one around this quadrilateral). Since the sum of internal angles of a convex quadrilateral is 2π, there must be a pair of opposite angles whose sum is at least π. Writing α_1, α_2, α_3, α_4 for the internal angles at the points A_1, A_2, A_3, A_4, respectively, assume, for example, that $\alpha_2 + \alpha_4 \ge \pi$. We will show that there is no closed disk that contains both A_1 and A_3 but neither of A_2 and A_4, showing that we can take $Y = \{A_1, A_3\}$ to witness the assertion.

Indeed, assume K is disk containing the points A_1 and A_3; we will show that then K contains at least one of the points A_2 or A_4. To see this, let A_1' and A_3' be the intersections of the boundary circle of this disk with the line $\overline{A_1A_3}$. Assume these points are labeled in such a way that A_1' is closer to A_1 and A_3' is closer to A_3. Clearly, A_1' and A_3' do not lie inside the linE segment $\overline{A_1A_3}$. Further, clearly, A_1', A_2, A_3', A_4, form a convex quadrilateral. Denoting the angles of this quadrilateral by α_1', α_2', α_3', α_4', we clearly have $\alpha_2' \ge \alpha_2$ and $\alpha_4' \ge \alpha_4$, and so $\alpha_2' + \alpha_4' \ge \pi$.

Let O be the center of the disk K, and let γ be the measure of the angle $\angle A_1'OA_3$ with $0 < \gamma < 2\pi$, where this angle is determined is such a way that its major part lies away from the point A_2.[2] Now, if $\alpha_2' \ge \gamma/2$, then A_2 will be contained in the disk K; if $\alpha_4' \ge \pi - \gamma/2$ then A_4 will be contained in the disk K.[3] As $\alpha_2' + \alpha_4' \ge \pi$, at least one of these inequalities will hold, and so at least one of the points A_3 or A_4 will belong to the disk K, as we wanted to show.

8) (SENIOR 4) The line

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contains ten pairs of parentheses and ten pairs of square brackets. One then proceeds to write one of the signs $<$ or $>$ inside each of the pairs of brackets in an arbitrary manner. Show that then it is always possible to write the integers 1, 2, ..., 10 inside the pairs of parentheses such that any two integers next to an inequality sign satisfy the inequality indicated.

Source: Problem 2 in a collection of Hungarian mathematics competition problems at
http://www.mek.iif.hu/porta/szint/termesz/matemat/matvers/html/III5.html

Solution: It is clear that it is always possible to write ten distinct rational numbers in the pairs parentheses in such a way that the inequalities are satisfied. In fact, all one needs to do is to write an arbitrary rational number inside the first pair; if the inequality to the right is $<$, then one writes a larger rational number inside the next pair, and if the inequality to the right is $>$, and one writes

[2]That is, if A_2 and O lies on the same side of the line $\overline{A_1A_3}$ then the angle as a region will include the line segment $\overline{A_1'A_3'}$, and if A_2 and O lie on the opposite sides, then this line segment will not be included in the region of the angle. In case O lies on the line, the angle region will be taken as the half plane not containing A_2.

[3]If a arc of a circle tends a angle γ with the center, then a point P on the circumference sees the chord determined by this arc with an angle $\gamma/2$. Any point P outside the circle will see the chord with an angle smaller than $\gamma/2$, and any point P inside the circle will see it with a larger angle. This is true for any angle $0 < \gamma < 2\pi$; it is assumed that the point P and the arc lie on the opposite half planes determined by the line connecting the endpoints of the arc.

a smaller rational number. One can proceed in this way from left to right, making sure that all the rational numbers are distinct. When finished, one can replace the smallest rational number with the integer 1, the next smallest one with the integer 2, the next one with the integer 3, and so on.

9) (SENIOR 5) Let $P(x)$, $Q(x)$, and $R(x)$ be polynomials that satisfy the identity

$$P(x^3) + xQ(x^3) = (1 + x + x^2)R(x).$$

Prove that $P(1) = Q(1) = R(1) = 0$.

Source: Problem 3.2.2, Journal: Function, Publisher: Monash University, Proposal: 3 (1979/2) 31, Solution: 3 (1979/5) 26. See

`http://www.problemcorner.org`

Solution: Consider the cubic root of unity

$$\alpha = -\frac{1}{2} + \frac{\sqrt{3}}{2}i,$$

where $i = \sqrt{-1}$ is the imaginary unit. We have

$$\alpha^2 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i,$$

and $\alpha^3 = 1$. In fact, α, α^2, and α^3 are the three roots of the equation $x^3 - 1 = 0$. Moreover, $x^3 - 1 = (x - 1)(x^2 + x + 1)$; α and α^2 are the zeros of the second factor, and α^3 is the zero of the first factor on the right-and side. Substituting α for x into the above identity for P, Q, and R, the right-hand side will be zero since α is a zero of $1 + x + x^2$; thus

$$P(1) + \alpha Q(1) = 0.$$

Similarly, substituting α^2 for x in the same identity, we obtain

$$P(1) + \alpha^2 Q(1) = 0.$$

These two equations can be solved for the unknowns $P(1)$ and $Q(1)$; we obtain $P(1) = Q(1) = 0$. There are several ways to see this; for example, one can point out that the matrix

$$\begin{pmatrix} 1 & \alpha \\ 1 & \alpha^2 \end{pmatrix}$$

is not singular, because its rows are linearly independent, or else, because its determinant

$$\begin{vmatrix} 1 & \alpha \\ 1 & \alpha^2 \end{vmatrix} = \alpha^2 - \alpha = -\sqrt{3}i$$

is not zero. Now, if we substitute $x = 1$ in the same identity, we obtain

$$P(1) + 1 \cdot Q(1) = 3R(1).$$

As we just saw, the left-hand side here is zero; hence we also have $R(1) = 0$. This completes the proof.

10) (SENIOR 6) Find the infinite sum

$$\frac{3}{0!} + \frac{4}{1!} + \frac{5}{2!} + \frac{6}{3!} + \ldots.$$

(The factorial function $n!$ is defined for nonnegative integers n by stipulating that $0! = 1$, and for $n > 0$, $n! = (n-1)! \cdot n$.)

Source: Based on Problem E109, American Mathematical Monthly 41 (1934), p. 447. See `http://www.problemcorner.org`

Solution: According to the Maclaurin expansion of e^x, we have

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!};$$

this series is convergent for every real number x. Multiplying both sides by x^3, we obtain

$$x^3 e^x = \sum_{n=0}^{\infty} \frac{x^{n+3}}{n!}.$$

Differentiating both sides, we obtain

$$(x^3 + 3x^2)e^x = \sum_{n=0}^{\infty} \frac{(n+3)x^{n+2}}{n!}.$$

Substituting $x = 1$, we now arrive at

$$4e = \sum_{n=0}^{\infty} \frac{n+3}{n!} = \frac{3}{0!} + \frac{4}{1!} + \frac{5}{2!} + \frac{6}{3!} + \ldots.$$

Thus, the value of the sum in question is $4e$.

Note. Alternatively, one can proceed as follows.

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{n+3}{n!} &= \sum_{n=0}^{\infty} \frac{n}{n!} + \sum_{n=0}^{\infty} \frac{3}{n!} = \sum_{n=1}^{\infty} \frac{n}{n!} + 3\sum_{n=0}^{\infty} \frac{1}{n!} \\ &= \sum_{n=1}^{\infty} \frac{n}{(n-1)! \cdot n} + 3\sum_{n=0}^{\infty} \frac{1}{n!} = \sum_{n=1}^{\infty} \frac{1}{(n-1)!} + 3\sum_{n=0}^{\infty} \frac{1}{n!}.\end{aligned}$$

For the second equation, note that in the first sum of the third member, we can start the summation with $n = 1$, since the term for $n = 0$ equals 0. In the first sum on the right-hand side, we can write $k = n - 1$, and then reintroduce n with the new meaning $n = k$:[4]

$$\sum_{n=1}^{\infty} \frac{1}{(n-1)!} = \sum_{k=0}^{\infty} \frac{1}{k!} = \sum_{n=0}^{\infty} \frac{1}{n!}.$$

[4] It is just more natural to use n for the summation index, since n is used everywhere else. Often, in a more compact writing style, one would suppress the sum involving k, and instead one would change the meaning of n without using the auxiliary variable k.

Since this sum equals e, and the second sum on the right-hand side above is the same, it follows that the sum to be evaluated is $4e$. The reason one would prefer the first method is that it is more transparent. For example, one can immediately see that

$$\sum_{n=0}^{\infty} \frac{(n+3)^2}{n!} = \frac{d}{dx} x \frac{d}{dx} x^3 e^x \bigg|_{x=1},$$

while it would be more time-consuming to evaluate the sum on the left by using the second method.[5]

11) (SENIOR 7) Assume f is a differentiable function on the real line such that

$$f(x)f(y) = f(x+y)$$

for all real x and y. Assuming that f is not identically 0, show that $f(x) = e^{cx}$ for some constant c.

Source: Problem 6 on p. 197 in Walter Rudin, *Principles of Mathematical Analysis*, Third Edition, McGraw-Hill, New York, 1976.

Solution: First, observe that $f(x) \neq 0$ for any real number x. Indeed, if $f(x) = 0$, then for any real y we have $f(x+y) = f(x)f(y) = 0 \cdot f(y) = 0$, and so f is identically 0, contradicting our assumption that this is not the case. Differentiating the equation $f(x)f(y) = f(x+y)$ with respect to y, for a fixed value of x, we obtain

$$f(x)f'(y) = f'(x+y).$$

Taking $y = 0$ and writing $f'(0) = c$, we have $f(x)c = f'(x)$, that is $f'(x)/f(x) = c$. Thus,[6]

$$\frac{f'(x)}{f(x)} = \frac{d}{dx} \log |f(x)| = c.$$

Hence $\log |f(x)| = cx + D$, i.e., $|f(x)| = e^{cx+D}$, that is $f(x) = \pm e^D \cdot e^{cx}$.

Note that whether to take $+$ or $-$ here does not depend on x, since f is continuous, and $f(x) \neq 0$ (as we remarked above), so f cannot change sign. Writing K for $\pm e^D$, this implies that $f(x) = Ke^{cx}$. Finally, the equation $f(x)f(y) = f(x+y)$ for $x = y = 0$ implies that $(f(0))^2 = f(0)$. This equation holds only if $f(0) = 0$ or $f(0) = 1$. since the former is not possible, as pointed out above, we must have $f(0) = 1$. Thus, the equation $f(x) = Ke^{cx}$ for $x = 0$ implies that $K = 1$, and so $f(x) = e^{cx}$, as we wanted to show.

Note. The same conclusion can also be established under the weaker assumption that f is continuous, without assuming that f is differentiable. To see this, first, observe (as in the proof above) that $f(x) \neq 0$ for any real number x. Indeed, if $f(x) = 0$, then for any real y we have $f(x+y) = f(x)f(y) = 0 \cdot f(y) = 0$, and so f is identically 0, contradicting our assumption that this is not the case. Further (as also pointed out above), the equation $f(x)f(y) = f(x+y)$ for

[5]The differential operator d/dx is interpreted as acting on everything that follows it. Thus, the use of parentheses on the right-hand side of the last formula is unnecessary. With the use of parentheses, the formula could be written as

$$\frac{d}{dx}\left(x \frac{d}{dx}\left(x^3 e^x\right)\right)\bigg|_{x=1}.$$

[6]$\log x$ denotes the natural logarithm of x; in mathematics, it is more common to use log than ln to denote the natural logarithm.

$x = y = 0$ implies that $(f(0))^2 = f(0)$. This equation holds only if $f(0) = 0$ or $f(0) = 1$. since the former is not possible, as pointed out above, we must have $f(0) = 1$. As f continuous and it is never 0, it cannot change sign in view of the Intermediate Value Theorem; hence we must have $f(x) > 0$ for all x.

Iterating the equation $f(x)f(y) = f(x + y)$ m times, we obtain that $f(nx) = (f(x))^m$ for any integer m positive integer m. Actually, it is also true for $m = 0$, since we have seen above that $f(0) = 1$. Furthermore, for an integer $m < 0$, we have

$$1 = f(0) = f(mx + (-m)x) = f(mx)f(-mx) = f(mx)(f(x))^{-m};$$

the last equation holds since $-m$ is a positive integer. Hence

$$f(mx) = \frac{1}{(f(x))^{-m}} = (f(x))^m$$

also in case $m < 0$. Next, for any integers m and n with $n \neq 0$ we have

$$f(mx) = f\left(n \cdot \frac{mx}{n}\right) = (f(mx/n))^n,$$

and so

$$f\left(\frac{m}{n}x\right) = (f(mx))^{1/n} = (f(x)^m)^{1/n} = (f(x))^{m/n};$$

for the first equation here, note that, as we pointed out above, the function f is always positive, so there is no problem with taking fractional powers. Thus, for any real x and rational r we have $f(rx) = (f(x))^r$. As f is continuous, for any real t we can approach t through a sequence of rational numbers to find that

$$f(tx) = \lim_{\substack{r \to t \\ r \text{ is rational}}} f(rx) = \lim_{\substack{r \to t \\ r \text{ is rational}}} (f(x))^r = (f(x))^t.$$

Hence $f(t) = f(t \cdot 1) = (f(1))^t$. Writing $c = \log f(1)$ (where log denotes natural logarithm), we have $f(1) = e^c$. thus,

$$f(t) = (f(1))^t = (e^c)^t = e^{ct}$$

for any real t, as we wanted to show.