Junior Prize Exam
Spring 2008

1) Prove that the sum of the squares of any five consecutive integers is divisible by 5.

2) Let f be a function defined for all real x, and suppose that

$$|f(x) - f(y)| \le (x - y)^2$$

for all real x and y. Prove that f is constant.

3) Prove that

$$(2^{2^0} + 1)(2^{2^1} + 1)(2^{2^2} + 1)(2^{2^3} + 1)\dots(2^{2^n} + 1) = 2^{2^{n+1}} - 1$$

holds for any integer $n \ge 0$.

4) Find all positive integers x and y such that $x^3 - y^3 = 91$.

5) Let m and n be positive integers such that $\sqrt{m} + \sqrt{n}$ is an integer. Prove that both $\sqrt{m}$ and $\sqrt{n}$ are integers.

6) How many ways can you pick two subsets A and B of the seven-element set $C = \{1, 2, 3, 4, 5, 6, 7\}$ such that the set $A \cap B$ is nonempty. **Note:** We want to count the ordered pairs (A, B); that is, we consider the choice $(A, B) = (\{3, 5, 7\}, \{5, 6\})$ different from the choice $(A, B) = (\{5, 6\}, \{3, 5, 7\})$.

7) Let A_1, A_2, A_3, A_4 be four distinct points in the plane and consider the set $X = \{A_1, A_2, A_3, A_4\}$. Prove that there exists a subset Y of X with the property that there is no closed disk K such that $K \cap X = Y$. (A closed disk is the set of points inside and on the circumference of a circle.)

Soon after the exam, solutions will appear on the Web Site
http://www.sci.brooklyn.cuny.edu/~mate/prize08/index.html

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