SENIOR PRIZE EXAM
SPRING 2008

1) Prove that the sum of the squares of any five consecutive integers is divisible by 5.

2) Let f be a function defined for all real x, and suppose that

$$|f(x) - f(y)| \leq (x - y)^2$$

for all real x and y. Prove that f is constant.

3) Prove that

$$(2^{2^0} + 1)(2^{2^1} + 1)(2^{2^2} + 1)(2^{2^3} + 1)\dots(2^{2^n} + 1) = 2^{2^{n+1}} - 1$$

holds for any integer $n \geq 0$.

4) The line

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contains ten pairs of parentheses and ten pairs of square brackets. One then proceeds to write one of the signs $<$ or $>$ inside each of the pairs of brackets in an arbitrary manner. Show that then it is always possible to write the integers 1, 2, ..., 10 inside the pairs of parentheses such that any two integers next to an inequality sign satisfy the inequality indicated.

5) Let $P(x)$, $Q(x)$, and $R(x)$ be polynomials that satisfy the identity

$$P(x^3) + xQ(x^3) = (1 + x + x^2)R(x).$$

Prove that $P(1) = Q(1) = R(1) = 0$.

6) Find the infinite sum

$$\frac{3}{0!} + \frac{4}{1!} + \frac{5}{2!} + \frac{6}{3!} + \dots.$$

(The factorial function $n!$ is defined for nonnegative integers n by stipulating that $0! = 1$, and for $n > 0$, $n! = (n-1)! \cdot n$.)

7) Assume f is a differentiable function on the real line such that

$$f(x)f(y) = f(x + y)$$

for all real x and y. Assuming that f is not identically 0, show that $f(x) = e^{cx}$ for some constant c.

SOON AFTER THE EXAM, SOLUTIONS WILL APPEAR ON THE WEB SITE
`http://www.sci.brooklyn.cuny.edu/~mate/prize08/index.html`

All computer processing for this manuscript was done under Fedora Linux. $\mathcal{A}\mathcal{M}\mathcal{S}$-TeX was used for typesetting.