All Problems on Prize Exam
Spring 2009
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The source for each problem is listed below when available; but even when the source is given, the formulation of the problem may have been changed. Solutions for the problems presented here were obtained without consulting sources for these solutions even when available, and additional information on how to solve these problems might be obtained by consulting the original sources. There was some overlap between the problems on the Junior and Senior prize exams; the problems common to both exams are listed only once.

1) (JUNIOR 1 and SENIOR 1) Show that the greatest number of lines that can be drawn in the plane in such a way that each line intersects exactly four of the other lines is eight.

Source: Norwegian Mathematical Olympiad, 1994

`http://abelkonkurransen.no/problems.php?lan=en`

Problem 19. Direct link: `http://abelkonkurransen.no/problems/abel_9394_r1_prob_en.pdf`

Solution: To get eight lines with this property, take four parallel lines in the plane, and then take four lines that are perpendicular to these.

To show that nine lines cannot have this property, assume we are given nine lines in the plane such that each intersects exactly four of the others. Take one of the lines; since this line only intersects four of the other lines, there must be four other lines parallel to it. If you take one of the lines that is not parallel to it, then this latter line will intersect all five of these parallel lines, showing that the required property is not fulfilled.

2) (JUNIOR 2 and SENIOR 2) Given four whole numbers a, b, c, and d, show that the product of the six differences $a-b$, $a-c$, $a-d$, $b-c$, $b-d$, $c-d$ is divisible by 12.

Source: József Kürschák High School Mathematics Competition (Kürschák József Matematikai Tanuló Verseny), Hungary,

`http://matek.fazekas.hu/list.php?what=competition`

1925, first problem. Direct link (in Hungarian):

`http://matek.fazekas.hu/show.php?problem=2747&setting=m`

Solution: Write P for the product. We need to show that P is divisible both by 3 and 4. As for divisibility by 3, at least two of the four numbers a, b, c, d must give the same remainder when divided by 3 (since there are only three available remainders: 0, 1, 2); their difference is divisible by 3.

As for divisibility by 4, there are two cases. First, there are three among the four numbers, say a, b, and c, that give the same remainder when divided by 2. Then each of the differences $a-b$, $a-c$, $b-c$ is divisible by 2, so P is divisible by 8 in this case. Second, two of the numbers, say a and b, give remainder 0 when divided by 2, and the other two, c and d, give remainder 1. In this case $a-b$ and $c-d$ are both divisible by 2, so P is again divisible by 4. This completes the proof.

3) (JUNIOR 3 and SENIOR 3) Color the points of the plane by two colors, say red and blue. Show that there will be two points of the same color exactly at one unit distance from each other.

Source:

`http://in.answers.yahoo.com/question/index?qid=20080125180827AAHtCUU`

All computer processing for this manuscript was done under Fedora Linux. $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX was used for typesetting.

Solution: Assume the assertion is not true. Pick an arbitrary point that is colored, say, blue. Then all points on the circumference of the circle of radius 1 with this point as its center must be colored red. Among these, there will be two exactly one unit distance apart.

4) (JUNIOR 4) Consider the regular decagon (10-sided polygon) inscribed into a circle, and consider a diagonal skipping two vertices of this decagon. Show that the difference between the length of this diagonal and the length of a side of the regular decagon equals the radius of the circle.

Source: József Kürschák High School Mathematics Competition (Kürschák József Matematikai Tanuló Verseny), Hungary,

`http://matek.fazekas.hu/list.php?what=competition`

1908, third problem. Direct link (in Hungarian):

`http://matek.fazekas.hu/show.php?problem=2707&setting=m`

Solution: While the result is not difficult to establish directly by using geometry, here we show a way of using complex roots of unity to obtain the result. Put[1]

$$\zeta = e^{2\pi i/10} = \cos\frac{2\pi}{10} + i\sin\frac{2\pi}{10}.$$

Put the vertices of the regular decagon inscribed in the unit circle $|z| = 1$ at at 1 ζ, ζ^2, ζ^3, ζ^4, ζ^4, ζ^6, ζ^7, ζ^8, ζ^9. One of the sides of this decagon connects the vertices ζ^3 and ζ^2, and the length of this side is $|\zeta^2 - \zeta^3| = \zeta^2 - \zeta^3$. Equality here holds because $\zeta^2 - \zeta^3$ is in fact a positive real number. It is easy to see this from a picture, but one can also verify this by calculation as follows.

For a complex number z, write $\bar{z}$ for its conjugate; we have $z\bar{z} = |z|^2$. Now $|\zeta| = 1$ and so $\bar{\zeta} = \zeta^{-1}$. We have $\zeta^5 = -1$ and $\zeta^{10} = 1$, and so $-\zeta^3 = \zeta^5 \cdot \zeta^3 = \zeta^8 = \zeta^{-2}\zeta^{10} = \zeta^{-2} = (\zeta^{-1})^2 = \bar{\zeta}^2 = \overline{\zeta^2}$. Hence $\zeta^2 - \zeta^3 = \zeta^2 + \overline{\zeta^2} = 2\Re\zeta^2$, where $\Re z = (z + \bar{z})/2$ indicates the real part of the complex number z. Hence $\zeta^2 - \zeta^3$ is indeed real; the reason $\Re\zeta^2$ is positive is that ζ^2 is in the first quadrant.

A diagonal skipping two vertices connects the points ζ^4 and ζ. Its length is $|\zeta - \zeta^4| = \zeta - \zeta^4$. Equality again holds since $\zeta - \zeta^4$ is a positive real number. Indeed, $-\zeta^4 = \zeta^5 \cdot \zeta^4 = \zeta^9 = \zeta^{-1} \cdot \zeta^{10} = \zeta^{-1} = \bar{\zeta}$, and so $\zeta - \zeta^4 = \zeta + \bar{\zeta} = 2\Re\zeta$; again, $\Re\zeta > 0$ since ζ is in the first quadrant.

Thus, in order to show that the difference between the length of the diagonal in question and that of a side equals 1 (the radius of the unit circle), we have to show that

$$(1) \qquad (\zeta - \zeta^4) - (\zeta^2 - \zeta^3) = 1.$$

i.e., that

$$-\zeta^4 + \zeta^3 - \zeta^2 + \zeta - 1 = 0.$$

Noting that $\zeta^5 = -1$, we have $-\zeta^4 = \zeta^5 \cdot \zeta^4 = \zeta^9$, $-\zeta^2 = \zeta^5 \cdot \zeta^2 = \zeta^7$, so this equation becomes $\zeta^9 + \zeta^7 + \zeta^5 + \zeta^3 + \zeta = 0$, or else $\zeta(\zeta^8 + \zeta^6 + \zeta^4 + \zeta^2 + 1) = 0$. As $\zeta \neq 0$, we can divide both sides by ζ:

$$\zeta^8 + \zeta^6 + \zeta^4 + \zeta^2 + 1 = 0.$$

To verify this, we may multiply both sides by $\zeta^2 - 1$, since $\zeta^2 - 1 \neq 0$. We obtain

$$\zeta^{10} - 1 = 0.$$

[1] Those not familiar with complex exponents may simply ignore the expression $e^{2\pi/10}$ and only read the trigonometric expression following it; this represents a point in the unit circle that results by dividing the unit circle into 10 equal parts. The fraction $2\pi/10$ can be reduced, at the price of clarity, though, since this way of writing it indicates that the circle is divided into 10 parts.

This being true, it follows that the earlier equations, and, in particular, equation (1), are also true.

Note. Here is an outline of a solution using elementary geometry. Let A, B, C, D, E, and F be six consecutive vertices of a regular decagon, inscribed into a circle with center O. The lines BE and CF intersect on the radius OD, for reasons of symmetry. Let G be the point of intersection. Also for reasons of symmetry, the opposite sides of the quadrilateral $CDEG$ are parallel, so $CDEG$ is a parallelogram. By a similar argument, $ABGO$ is also a parallelogram. Therefore,

$$BE - CD = BE - GE = BG = AO,$$

which is what we wanted to prove.

5) (JUNIOR 5) A society created to help the police contains exactly 100 persons. Every evening three persons are on duty. Prove that one cannot organize duties in such a way that every couple will meet on duty exactly once (during a certain time period).

Source: The All-Soviet-Union mathemtics competitions, 1961-1987,
`http://www.mathprocess.com/archive/RusMath.txt`
Problem 061

Solution: Let the persons be numbered 0, 1, 2, ..., 99. Person 0 must meet each of persons 1, 2, ... 99. Since on each team, person 0 can meet only two other persons, there must be at least 50 teams containing person 0 so he/she can meet the other 99 persons. But then, there is a person among persons 1, 2, ..., 99 that is on at least two of these teams. Person 0 will then meet this person twice.

6) (JUNIOR 6) Find a function $f(x)$ defined for $x > 1$ such that

$$\int_x^{x^2} f(t)\,dt = 1$$

for all $x > 1$.

Source: 101 Mathematical Problems, complied by Yang Wang, second problem set,
`http://www.math.gatech.edu/~wang/Putnam/second100.pdf`
Problem 83.

Solution: We will not try to determine all functions satisfying the conditions; in fact, we will impose additional conditions on f to make such an f easier to find. First, we assume that f is continuous on $(1, +\infty)$, so we can use the Fundamental Theorem of Calculus to differentiate the integral. Second, we will strengthen the assumption to require that, for any real α and any $x > 1$ the integral

$$\int_x^{x^\alpha} f(t)\,dt = 1$$

does not depend on x.[2] Differentiating this equation with respect to x, we obtain with any $c > 1$ that

$$0 = \frac{d}{dx}\int_x^{x^\alpha} f(t)\,dt = \frac{d}{dx}\int_c^{x^\alpha} f(t)\,dt - \frac{d}{dx}\int_c^{x} f(t)\,dt = \alpha x^{\alpha-1} f(x^\alpha) - f(x),$$

[2]This assumption will only imply that

$$\int_x^{x^2} f(t)\,dt$$

is constant, but then f can be multiplied by an appropriate number to make sure that this constant equals 1.

where, to obtain the last equation, we used the chain rule and the Fundamental Theorem of Calculus. Putting $f(e) = 1$ where e is the base of the natural logarithm, and writing $x = e$, this equation gives

$$f(e^\alpha) = \frac{1}{\alpha e^{\alpha-1}} = \frac{e}{\alpha e^\alpha}.$$

Writing $t = e^\alpha$, this gives

$$f(t) = \frac{e}{t \log t}.$$

Now, with this f, for any $x > 1$ we have

$$\int_x^{x^2} f(t)\,dt = \int_x^{x^2} \frac{e\,dt}{t \log t} = \int_{\log x}^{2\log x} \frac{e\,du}{u} = e \log 2,$$

where the second equality was obtained by making the substitution $u = \log t$ ($du = dt/t$).

So if instead of the above choice for f, we take a constant multiple, namely

$$f(t) = \frac{1}{\log 2} \cdot \frac{1}{t \log t},$$

then the equation

$$\int_x^{x^2} f(t)\,dt = 1$$

will be satisfied for any $x > 1$.

7) (JUNIOR 7) On an infinite chess board, each square is marked with an arrow pointing in one of the eight directions of $0°$, $\pm 45°$, $\pm 90°$, $\pm 135°$, and $180°$ (negative angles mean counterclockwise turns), so each square has an arrow pointing to one of its eight nearest neighbors. The arrows on squares sharing an edge differ by at most $45°$ (multiples of $360°$ are ignored here, so the angle between the arrow pointing in direction $180°$ and $-135°$ is considered to be $45°$). A king is placed randomly on one of the squares, and it moves from square to square following the arrows. Prove that the king will never get back to its starting square.

Source: Stan Wagon's problem list,

`http://mathforum.org/wagon/spring98/lemming.html`

Original source: Ravi Vakil, *A Mathematical Mosaic: Patterns & Problem Solving* (Brendan Kelly Publishing, 1996).

Solution: Call a pair (s_1, s_2) an edge if (s_1, s_2) are adjacent squares, i.e., squares sharing a common edge. Write $-(s_1, s_2) = (s_2, s_1)$. Consider the set V of all formal sums

$$\sum_k c_k \cdot (s_k, t_k)$$

where c_k is an integer (positive, negative, or zero), and (s_k, t_k) is an edge; only sums containing finitely many terms are considered. (V is a *module*[3] over the set $\mathbb{Z}$ of integers.) To make the formal

[3]A module is a generalization of vector space, where the scalars are assumed to be only a ring rather than a field. See

`http://en.wikipedia.org/wiki/Module_(mathematics)` arguments simpler, one may also consider the edge (s, s) containing the same square twice, and then take $(s, s) = 0$.[4] Call a sum

$$\sum_{k=1}^{n-1} 1 \cdot (s_k, s_{k+1})$$

a circuit if $s_n = s_1$. (The coefficient 1 will be omitted below; also terms with zero coefficients may be omitted.) For such circuit, we can assign the total turn of the arrow by adding the angles between the squares s_k and s_{k+1}, each of these angles being $0°$ or $\pm 45°$.

Call such a circuit a C simple if no square occurs more than once among $s_1, s_2, \ldots, s_{n-1}$. Clearly, every circuit is a sum of simple circuits. In fact, if $s_i = s_j$ for some i, j with $1 \le i < j \le n$ then

$$C = \sum_{k:(1<k\le i)\vee(j\le k<n)} (s_k, s_{k+1}) + \sum_{k:i\le k<j} (s_k, s_{k+1}),$$

and each of these sums is a circuit.[5]

We claim that the total turn of the arrow for a simple circuit C is $0°$. This is because such a circuit C can be represented as a sum of circuits

$$(t_1, t_2) + (t_2, t_3) + (t_3, t_4) + (t_4, t_1)$$

of length four. In fact, if the simple circuit is directed counterclockwise, take all squares in this circuit and all the squares inside the region surrounded by the circuit; form all groups of four adjacent squares among these squares, make them into circuits directed counterclockwise. Then C will be the sum of all these four-element circuits. The total turn of the arrow for C will be the sum of the turns of the arrow for each of these for four-element circuits. Since the total turn of arrow for each of these four-element circuits is $0°$, the same will also be true for C.

Since every circuit can be written as a sum of simple circuits, it is also true that the total turn of the arrow for any circuit is $0°$.

Assume that the king started on a square and then got back to the same square the first time. Connect the centers of the squares along which the king moved to form a closed path consisting of straight line segments. Notice that this closed path cannot intersect itself. Assume, on the contrary, that there is such a self-intersection. This cannot involve a square the king traverses twice, since then the king would have to continue in the same direction both times when following the arrow. The only other kind of self-intersection that might be possible is when

$$(s_1, s_2) + (s_2, s_3) + (s_3, s_4) + (s_4, s_1)$$

is a circuit of length four, and the king first traverses the diagonal s_1 to s_3, and then the diagonal s_2 to s_4; but this is not possible, since arrows on squares s_1 and s_2 form an angle $0°$ or $\pm 45°$. So by adding up the turns of the arrow for each move of the king (each of these turns being $0°$, $\pm 45°$, or $\pm 90°$), the arrow must have made a total turn of either $360°$ or $-360°$ during the king's trip. Now, to make the king's trip into a circuit, for each diagonal move one must add one of the squares adjacent to both of the diagonally touching squares. For this circuit, the total turn of the arrow

[4] This latter 0 is, of course, denoting the zero element of the module V, which is technically different from the number 0. If one is really fastidious, one could write a bold face $\mathbf{0}$ for this.

[5] Unless the first sum is the empty sum in the trivial case $i = 1$ and $j = n$.

will still be $\pm 360°$. This is a contradiction, since we saw that the total turn of the arrow for every circuit is $0°$.

8) (SENIOR 4) Assume f is twice differentiable on $(0, +\infty)$, f'' is bounded on $(0, +\infty)$, and $\lim_{x\to+\infty} f(x) = 0$. Show that $\lim_{x\to+\infty} f'(x) = 0$.

Source: Walter Rudin, *Principles of Mathematical Analysis, third edition*, McGraw-Hill, New York, 1976. Chapter 5, Exercise 16, p. 116.

Solution: Let $a, x > 0$. According to Taylor's formula with the Lagrange remainder term, we have

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(\xi)}{2}(x-a)^2$$

for some ξ between a and x. Let $\epsilon > 0$ be arbitrary, and let $x = a + \epsilon$. Then

$$f(a+\epsilon) = f(a) + f'(a)\epsilon + \frac{f''(\xi_a)}{2}\epsilon^2$$

for some $\xi_a \in (a, a+\epsilon)$, i.e.,

$$f'(a)\epsilon = f(a+\epsilon) - f(a) - f''(\xi_a)\epsilon^2/2.$$

The assumption says that f'' is bounded, say $|f''(x)| \leq M$ for every $x > 0$. Then it follows that

$$|f'(a)|\epsilon = |f(a+\epsilon) - f(a)| + M\epsilon^2/2.$$

Making $a \to +\infty$, it follows that[6]

$$\limsup_{a\to\infty} |f'(a)|\epsilon \leq M\epsilon^2/2,$$

that is

$$\epsilon \limsup_{a\to\infty} |f'(a)| \leq M\epsilon^2/2.$$

We can divide both sides by ϵ to obtain

$$\limsup_{a\to\infty} |f'(a)| \leq M\epsilon/2.$$

Since this is true for every $\epsilon > 0$, it follows that

$$\limsup_{a\to\infty} |f'(a)| = 0,$$

[6] For a function g, its upper limit, or *limit superior*, at $+\infty$ is defined as

$$\limsup_{x\to+\infty} g(x) = \lim_{x\to+\infty} (\sup\{g(t) : t \geq x\}).$$

If g is bounded on $(0, +\infty)$ then the limit on the right exists, since in this case

$$\sup\{g(t) : t \geq x\}$$

is a bounded nonincreasing (i.e., decreasing in the wider sense) function of x.

The use of limit superior in this proof is easily avoided at the price of minor complications, and it is an interesting exercise to transcribe the proof in a way that does not make use limit superior.

i.e.,

$$\lim_{a\to\infty} f'(a) = 0,$$

which is what we wanted to show.

9) (SENIOR 5) Let S be a set of 16 distinct integers, each greater than or equal to 1 and less than or equal to 30. Show that there must exist two elements in S which differ by exactly 3.

Source: Math 199, Problem Solving and Putnam Preparation, Spring 2000 (Geoff Mess, Terry Tao, Christoph Thiele),

`http://www.math.ucla.edu/~tao/putnam/`

fifth assignment.

Solution: Consider the sets $S_i = \{i, i+3\}$ for each $i \in S$. Assuming that there are no elements with difference 3 in S, these sets must be pairwise disjoint. Writing

$$M = \{1, 2, , 3, \dots, 32, 33\},$$

$S_i \subset S$ for each $i \in S$. Since M has 33 elements, it can have at most 16 pairwise disjoint two-element subsets; this shows that S cannot have more than 16 elements. This, however, does not lead to a contradiction, since in fact S has exactly 16 elements, so this argument must be sharpened.

Write

$$M_j = \{n : 1 \le n \le 33 \text{ and } n = 3k + j \text{ for some integer } j\}.$$

Then, for each $i \in S$, we have $S_i \subset M_j$ for $j = 0$, $j = 1$, or $j = 2$. Since M_j has 11 elements, at most 5 of these S_i's can be a subset of M_j, for each j. This allows only for a number $3 \cdot 5$ of S_i's, showing that S having 16 elements is impossible.

Note on grammar. In English, "'s" indicates genitive, and not plural; when forming a plural, no apostrophe is used. This is, however, unsatisfactory when forming plurals of mathematical symbols. For example, the apostrophe in the plural "S_i's" is used to separate mathematical symbols from text. This convention is used at least by some mathematical writers.

10) (SENIOR 6) For every real number x_1, construct the sequence x_1, x_2, ... by setting

$$x_{n+1} = x_n \left(x_n + \frac{1}{n} \right)$$

for each $n \ge 1$. Prove that there exists exactly one value of x_1 for which

$$0 < x_n < x_{n+1} < 1$$

for every n.

Source: Twenty-sixth International Mathematical Olympiad, 1985, Problem 6,

`http://imo.math.ca/Exams/1985imo.html`

Solution: Let t be a positive real number, and define the functions $x_n(t)$ by putting $x_1(t) = t$ and

$$x_{n+1}(t) = x_n(t) \left(x_n(t) + \frac{1}{n} \right) = x_n^2(t) + \frac{x_n}{n}; \tag{1}$$

this is exactly the same as the definition of the sequence x_n above, except that here we want to emphasize that $x_n(t)$ are functions of the first member $x_1(t) = t$. The following simple observations can be made: (i) We have $x_n(t) > 0$ for all $n \geq 1$ for every t. (ii) $x_n(t)$ is a differentiable function of t, and its derivative is continuous. (iii) The derivative $x'_n(t)$ is positive for every n and t. This is easily established by induction, since

$$x'_{n+1}(t) = 2x_n(t)x'_n(t) + \frac{x'_n(t)}{n}.$$

This equation also shows that (iv) $x'_{n+1}(t) > x'_n(t)$ whenever $x_n(t) \geq 1/2$.

It is now easy to show the uniqueness of x_1 in the problem. Assume, on the contrary, that there are t_1 and t_2 with $t_1 < t_2$ such that

$$0 < x_n(t_i) < x_{n+1}(t_i) < 1$$

for every $n \geq 1$ and for $i = 1, 2$. The inequality $x_n(t_i) < x_{n+1}(t_i)$ implies that $x_n(t_i) > 1 - 1/n$, and so

$$\lim_{n\to\infty} x_n(t_i) = 1. \tag{2}$$

As $x_n(t)$ is an increasing function of t (its derivative being positive), we also have $x_n(t) > 1 - 1/n$ for every t with $t_1 \leq t \leq t_2$. Hence $x_n(t) > 1/2$ for $n > 2$, and so $x'_n(t) > x'_2(t) > 0$; thus, there is and $\epsilon > 0$ such that $x'_n(t) > \epsilon$ for every $n \geq 2$ and every t with $t_1 \leq t \leq t_2$. Therefore, using the Mean-Value Theorem of differentiation, given $n \geq 2$, there is a $\xi \in (t_1, t_2)$ such that

$$x_n(t_2) - x_n(t_1) = (t_2 - t_1)x'_n(\xi) > (t_2 - t_1)\epsilon.$$

This contradicts (2) above.

To show the existence of an x_1 as required, write

$$S = \{t > 0 : x_n(t) \leq 1 \text{ for all integers } n\}.$$

The set S is nonempty. This can be seen by noting that if $t > 1$ is such that for some $n > 1$ we have $x_k(t) < 1$ for $1 \leq k < n$ and $x_n(t) < 1 - 1/n$, then $t \in S$. Indeed, we have $x_n(t) > x_{n+1}(t) > x_{n+2}(t) > \dots$ by (1), and so we in fact have $t \in S$. This shows that $1/4 \in S$, because $x_2(1/4) = 5/16 < 1 - 1/2$. Further, S is bounded from above, since no $t > 1$ belongs to S.

Write $s = \sup S$. We claim that then the sequence $x_n = x_n(s)$ satisfies the requirements of the problem. To see this, first notice that we have $x_n(s) \leq 1$ for every $n \geq 1$; this is because $x_n(t)$ is a continuous function of t, and $x_n(t) \leq 1$ for every $t \in S$. Second, notice that we cannot have $x_n(s) < 1 - 1/n$ for any $n > 1$. Indeed, if this were the case, then, in view of the continuity of $x_n(t)$, there would be a $t > s$ such that $x_n(t) < 1 - 1/n$. Then $t \in S$. Indeed, for $k < n$, we have $x_k(t) < 1$ (if we have $x_k(t) \geq 1$ then we would also have $x_l(t) \geq 1$ for every $l > k$ according to (1)); further, we also have $x_k(t) < x_n(t) < 1 - 1/n < 1 - 1/k$ for $k > n$ according to (1).[7] The relation $t \in S$ contradicts the choice that s is the supremum of S.

Note also that we cannot even have $x_n(s) = 1 - 1/n$, since, if this were the case, we would have $x_{n+1}(s) = x_n(s) = 1 - 1/n < 1 - 1/(n+1)$, which, as we just saw, is impossible. Therefore, $1 - 1/n < x_n(s) \leq 1$ for every $n \geq 1$. Hence, (1) implies that

$$0 < x_n(s) < x_{n+1}(s) \leq 1$$

[7] One needs to use a simple induction on k to show this.

according to (1). The last inequality is actually strict, since the above inequality implies that $x_n(s) < 1$ for every $n > 1$. This completes the proof of the existence of an appropriate x_1 (with $x_1 = s$).

11) (SENIOR 7) Let $a_2, a_3, \ldots$ be a sequence of positive real numbers such that the series $\sum_{n=2}^{\infty} a_n$ is convergent. Show that the series $\sum_{n=2}^{\infty} a_n^{1-1/\ln n}$ is also convergent.

Source: Középiskolai Matematikai és Fizikai Lapok (Hungarian High School Mathematics and Physics Journal),

http://www.komal.hu/info/bemutatkozas.e.shtml

Problem N. 150, October 1997 (in Hungarian). Direct link:

http://www.komal.hu/verseny/1997-10/mat.h.shtml

Solution: We will start the above sums with $n = 3$ instead of $n = 2$; this will not affect convergence.[8]

$$S = \{n \geq 3 : a_n \leq 1/n^2\} \quad \text{and} \quad T = \{n \geq 3 : a_n > 1/n^2\}.$$

Given any $n \geq 3$, the derivative of the function $x^{1-1/\ln n}$ for $x > 0$ is $(1 - 1/\ln n)x^{-1/\ln n}$, which is positive; hence this function is an increasing function of x. Therefore, for $n \in S$ we have[9]

$$a_n^{1-1/\ln n} \leq \left(\frac{1}{n^2}\right)^{1-1/\ln n} = \frac{1}{n^2} \cdot \left(\frac{1}{n^2}\right)^{-1/\ln n} = \frac{1}{n^2} \exp\left(-\frac{1}{\ln n} \cdot \ln \frac{1}{n^2}\right) = \frac{e^2}{n^2}.$$

Hence

$$\sum_{n \in S} a_n^{1-1/\ln n} \leq \sum_{n=3}^{\infty} \frac{e^2}{n^2}$$

is convergent.

For $n \in T$ we have $1/a_n < n^2$, and so $\ln(1/a_n) < \ln n^2$. Noting that the function e^x is increasing, we therefore have

$$\begin{aligned} a_n^{1-1/\ln n} &= a_n \cdot a_n^{-1/\ln n} = a_n \exp\left(-\frac{1}{\ln n} \cdot \ln a_n\right) \\ &= a_n \exp\left(\frac{1}{\ln n} \cdot \ln \frac{1}{a_n}\right) < a_n \exp\left(\frac{1}{\ln n} \cdot \ln n^2\right) = e^2 a_n. \end{aligned}$$

Hence

$$\sum_{n \in T} a_n^{1-1/\ln n} \leq \sum_{n \in T} e^2 a_n$$

is also convergent.[10] This establishes the assertion.

[8] The reason for considering $n \geq 3$ only is that $1 - 1/\ln n > 0$ for $n \geq 3$.

[9] To avoid writing tiny letters in the exponent, we will write $\exp(x)$ for e^x.

[10] The reason for writing $\leq$ instead of $<$ in the displayed line above is that the set T may be empty, in which case both sides are zero.