

JUNIOR PRIZE EXAM
SPRING 2009

- 1) Show that the greatest number of lines that can be drawn in the plane in such a way that each line intersects exactly four of the other lines is eight.
- 2) Given four whole numbers a , b , c , and d , show that the product of the six differences $a - b$, $a - c$, $a - d$, $b - c$, $b - d$, $c - d$ is divisible by 12.
- 3) Color the points of the plane by two colors, say red and blue. Show that there will be two points of the same color exactly at one unit distance from each other.
- 4) Consider the regular decagon (10-sided polygon) inscribed into a circle, and consider a diagonal skipping two vertices of this decagon. Show that the difference between the length of this diagonal and the length of a side of the regular decagon equals the radius of the circle.
- 5) A society created to help the police contains exactly 100 persons. Every evening three persons are on duty. Prove that one cannot organize duties in such a way that every couple will meet on duty exactly once (during a certain time period).
- 6) Find a function $f(x)$ defined for $x > 1$ such that

$$\int_x^{x^2} f(t) dt = 1$$

for all $x > 1$.

- 7) On an infinite chess board, each square is marked with an arrow pointing in one of the eight directions of 0° , $\pm 45^\circ$, $\pm 90^\circ$, $\pm 135^\circ$, and 180° (negative angles mean counterclockwise turns), so each square has an arrow pointing to one of its eight nearest neighbors. The arrows on squares sharing an edge differ by at most 45° (multiples of 360° are ignored here, so the angle between the arrow pointing in direction 180° and -135° is considered to be 45°). A king is placed randomly on one of the squares, and it moves from square to square following the arrows. Prove that the king will never get back to its starting square.

SOON AFTER THE EXAM, SOLUTIONS WILL APPEAR ON THE WEB SITE

<http://www.sci.brooklyn.cuny.edu/~mate/prize09/index.html>

All computer processing for this manuscript was done under Fedora Linux. $\mathcal{A}\mathcal{M}\mathcal{S}$ - $\mathcal{T}\mathcal{E}\mathcal{X}$ was used for typesetting.