SENIOR PRIZE EXAM
SPRING 2009

1) Show that the greatest number of lines that can be drawn in the plane in such a way that each line intersects exactly four of the other lines is eight.

2) Given four whole numbers a, b, c, and d, show that the product of the six differences $a - b$, $a - c$, $a - d$, $b - c$, $b - d$, $c - d$ is divisible by 12.

3) Color the points of the plane by two colors, say red and blue. Show that there will be two points of the same color exactly at one unit distance from each other.

4) Assume f is twice differentiable on $(0, +\infty)$, f'' is bounded on $(0, +\infty)$, and $\lim_{x \to +\infty} f(x) = 0$. Show that $\lim_{x \to +\infty} f'(x) = 0$.

5) Let S be a set of 16 distinct integers, each greater than or equal to 1 and less than or equal to 30. Show that there must exist two elements in S which differ by exactly 3.

6) For every real number x_1, construct the sequence x_1, x_2, ... by setting

$$x_{n+1} = x_n \left(x_n + \frac{1}{n} \right)$$

for each $n \geq 1$. Prove that there exists exactly one value of x_1 for which

$$0 < x_n < x_{n+1} < 1$$

for every n.

7) Let a_2, a_3, ... be a sequence of positive real numbers such that the series $\sum_{n=2}^{\infty} a_n$ is convergent. Show that the series $\sum_{n=2}^{\infty} a_n^{1 - 1/\ln n}$ is also convergent.

SOON AFTER THE EXAM, SOLUTIONS WILL APPEAR ON THE WEB SITE
http://www.sci.brooklyn.cuny.edu/~mate/prize09/index.html