All Problems on Prize Exam
Spring 2010
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The source for each problem is listed below when available; but even when the source is given, the formulation of the problem may have been changed. Solutions for the problems presented here were obtained without consulting sources for these solutions even when available, and additional information on how to solve these problems might be obtained by consulting the original sources. There was some overlap between the problems on the Junior and Senior prize exams; the problems common to both exams are listed only once.

1) (JUNIOR 1 and SENIOR 1) Show that there is no integer whose square, written in the decimal system, ends in two odd digits.

Source: Középiskolai Matematikai és Fizikai Lapok (Hungarian Mathematics and Physics Journal for High Schools), Problem C. 675, Vol. 52 (2002/5), p. 294.

Solution: Assume $n = 10k + l$, where $k \ge 0$ and l with $0 \le l \le 9$ is such an integer. We have

$$n^2 = 100k^2 + 20kl + l^2.$$

Hence, the last two digits of n^2 are odd if and only if both digits of l^2 are odd. For this, l must be odd; but $1^2 = 01$, $3^2 = 09$, $5^2 = 25$, 49, and $9^2 = 81$, so this does not happen for any l. The proof is complete.

2) (JUNIOR 2 and SENIOR 2) Let n be an arbitrary positive integer. Show that

$$n(n+2)(5n-1)(5n+1)$$

is divisible by 24.

Source: Matematikai Lapok, 1904/04, exercise 427, see

`http://db.komal.hu/scan/1904/04/90404148.g4.png`

For the whole archive, see

`http://www.komal.hu/lap/archivum.h.shtml`

Solution: The assertion is true with any integer n, that is, n does not need to be positive. We need to show that the expression is divisible by 8 and by 3. To show divisibility by 8, note that if n is even, then one of n and $n+2$ is divisible by 4 (and the other is divisible by 2), so $n(n+2)$ is divisible by 8. If n is odd, then one of $5n-1$ and $5n+1$ is divisible by 4 (and the other is divisible by 2). Then $(5n-1)(5n+1)$ is divisible by 8.

As for divisibilty by 3, we have

$$n(n+2)(5n-1)(5n+1) = -n\big((n+1) - 6n\big)(n+2)(5n+1),$$

and this is divisible by 3 exactly when

$$n(n+1)(n+2)(5n+1)$$

is divisible by 3. Among the numbers n, $n+1$, $n+2$, one must be divisible by 3, showing that the original expression is also divisible by 3.

All computer processing for this manuscript was done under Fedora Linux. $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX was used for typesetting.

3) (JUNIOR 3 and SENIOR 3) Show that there is no triangle whose altitudes are of length 4, 7, and 10 units.

Source: International Mathematical Talent Search – Round 8, Problem 1/8, coordinator Dr. George Berzsenyi

`http://www.cms.math.ca/Competitions/IMTS/`

and

`http://www.cms.math.ca/Competitions/IMTS/imts8.html`

Solution: Assume there is such a triangle. Let the sides of the triangle be a, b, and c, and let the corresponding altitudes be 4, 7, and 10. Writing A for the area of the triangle, we then have

$$2A = 4a = 7b = 10c,$$

that is

$$b = \frac{4}{7}a \quad \text{and} \quad c = \frac{4}{10}a = \frac{2}{5}a.$$

Then

$$b + c = \left(\frac{4}{7} + \frac{2}{5}\right) a = \frac{34}{35}a < a.$$

This is a contradiction, since in any triangle, the sum of two sides are greater then the third side.

4) (JUNIOR 4) Show that

$$\sqrt[3]{\sqrt{5}+2} - \sqrt[3]{\sqrt{5}-2} = 1.$$

Source: Középiskolai Matematikai és Fizikai Lapok (Hungarian Mathematics and Physics Journal for High Schools), Problem B. 3623, Vol. 53 (2003/3), p. 156.

Solution: Writing

$$r = \sqrt[3]{\sqrt{5}+2} - \sqrt[3]{\sqrt{5}-2},$$

$a = \sqrt{5} + 2$, and $b = \sqrt{5} - 2$, we have

$$r^3 = \left(\sqrt[3]{a} - \sqrt[3]{b}\right)^3 = a - 3\sqrt[3]{a^2b} + 3\sqrt[3]{ab^2} - b.$$

An easy calculation shows that

$$a^2b = (\sqrt{5}+2)^2(\sqrt{5}-2) = (\sqrt{5}+2)\,(\sqrt{5}+2)(\sqrt{5}-2) = (\sqrt{5}+2)\,(5-4) = \sqrt{5}+2 = a.$$

Similarly

$$ab^2 = (\sqrt{5}+2)(\sqrt{5}-2)^2 = (\sqrt{5}+2)(\sqrt{5}-2)\,(\sqrt{5}-2) = (5-4)\,(\sqrt{5}-2) = \sqrt{5}-2 = b.$$

Hence

$$r^3 = a - 3\sqrt[3]{a} + 3\sqrt[3]{b} - b = a - b - 3r = 4 - 3r.$$

That is,

$$r^3 + 3r - 4 = 0.$$

It is immediate that $r = 1$ is a solution of this equation. Now

$$r^3 + 3r - 4 = (r-1)(r^2 + r + 4),$$

and the equation $r^3 + r + 4 = 0$ has no real solutions. As r is real, we must have $r = 1$, as we wanted to show.

5) (JUNIOR 5) How many nonnegative real solutions does the equation $x = 2000 \sin x$ have (of course, we are using radians here when evaluating $\sin x$). (**Note:** The approximation of π up to five decimal places is 3.14159.)

Source: Missouri State University Problem Archive, Advanced Problem #14,

`http://faculty.missouristate.edu/l/lesreid/Adv14.html`

For more problems at the site, see

`http://faculty.missouristate.edu/l/lesreid/ADVarchives.html`

Solution: As $-2000 \le 2000 \sin x \le 2000$, all nonnegative solutions lie in the interval $[0, 2000]$. Let k be a nonnegative integer. In the interval $((2k+1)\pi, (2k+2)\pi)$ the equation has no solution, since $\sin x$ is negative there. In the interval $[2k\pi, (2k+1)\pi]$ there may be 0, 1, or 2 solutions. The number of solutions will be 0 is $x > 2000$, since then the left-hand side is greater than the right-hand side.

We have $\sin 2k\pi = \sin(2k+1)\pi = 0$ and $\sin(2k+1/2)\pi = 1$. Noting that $\sin x$ is concave[1] in the interval $[2k\pi, (2k+1)\pi]$, the equation has exactly two solutions in case $(2k+1)\pi \le 2000$.

The case $2k\pi < 2000 < (2k+1)\pi$ is somewhat more complicated, because it might happen that the curve $y = x$ is tangent to the curve $y = 2000 \sin x$, in which case the equation has a double solution, and this may be counted as a single solution, depending on how one interprets the question in the problem. Fortunately, this problem will not come up. This is because, from the decimal representation of π one can see that $2k\pi < 2000 < (2k+1)\pi$ happens exactly for $k = 318$. For this k, we have $(2k+1/2)\pi < 2000$,[2] so, by the Intermediate Value Theorem, the equation must have one solution in each the intervals $(2k\pi, (2k+1/2)\pi)$ and $(2k+1/2)\pi, (2k+1)\pi)$, so the two solutions do not coincide. That is, for each value of $k = 0, 1, 2, \ldots, 318$ we get exactly two solutions, giving altogether $2 \cdot 319 = 638$ solutions.[3]

6) (JUNIOR 6) If $r + s + t = 3$, $r^2 + s^2 + t^2 = 1$ and $r^3 + s^3 + t^3 = 3$, compute rst.

Source: 2007 Rice University Math Tournament, Algebra Test, Problem 8.

`http://www.ruf.rice.edu/~eulers/tests/2007test/alg2007.pdf`

Solution: We want to determine the coefficients of rst, r^2s, and r^3 in $(r+s+t)^3$. To this end, write $r = r_1 = r_2 = r_3$, $s = s_1 = s_2 = s_3$, $t = t_1 = t_2 = t_3$. Then

$$(r + s + t)^3 = (r_1 + s_1 + t_1)(r_2 + s_2 + t_2)(r_3 + s_3 + t_3).$$

The only way we get r^3 here is $r_1r_2r_3$, i.e., the coefficient of r^3 is 1. We can get r^2s as $r_1r_2s_3$, $r_1r_3s_2$, and $r_2r_3s_1$, i.e., the coefficient of r^2s is 3. The coefficient of rs^2 is the same, for reasons of symmetry. We can get rst as $r_1s_2t_3$, $r_3s_1t_3$, $\ldots$, i.e., six ways, associated with all six permutations

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[1] A function is called *concave* if it is concave down, and it is called *convex* if it is concave up. However, the terms "concave up" and "concave down" are only used in college courses on calculus, and are not used in mathematics. For that matter, the term "calculus" is not used in mathematics to describe the material taught in college calculus courses; the term that is used is "mathematical analysis". The term originates in l'Hospital's 1696 book Analyse des infiniment petits pour l'intelligence des lignes courbes (Analysis of the Infinitely Small to Learn about Curved Lines).

[2] We have $3.14 < \pi < 3.142$, and, with $k = 318$, $(2k+1/2)\pi < (2k+1/2) \cdot 3.142 = 1999.883$ and $(2k+1)\pi > (2k+1) \cdot 3.14 = 2000.18$ (the last two equalities involving decimals are exact, and not approximate, equalities).

[3] Note that no solution is double-counted here, since a solution cannot be at the endpoint of any of these intervals (except the solution at $x = 0$), since π is irrational.

of 123. Furthermore, in

$$(r^2+s^2+t^2)(r+s+t) == (r_1^2+s_1^2+t_1^2)(r_2+s_2+t_2) =$$

we have r^3 occur as $r_1^2 r_2$, i.e. r^3 occurs once. Further, $r^2 s$ occurs $r_1^2 s_2$ occurs once. The occurrence of other terms can be inferred. Therefore,

$$(r+s+t)^3 = -2(r^3+s^3+t^3) + 3(r^2+s^2+t^2)(r+s+t) + 6rst.$$

Substituting the given numerical values here, we obtain

$$3^3 = -2 \cdot 3 + 3 \cdot 1 + 6rst.$$

This gives $rst = 5$.

7) (JUNIOR 7) Let n be a positive integer, and assume we are given $2n-1$ irrational numbers. Prove that it is possible to select n numbers $x_1, x_2, \ldots, x_n$ among them such that, given arbitrary nonnegative rational numbers $a_1, a_2, \ldots, a_n$, the number

$$\sum_{i=1}^{n} a_i x_i$$

is rational if and only if $a_1 = a_2 = \ldots = a_n = 0$.

Source: Középiskolai Matematikai és Fizikai Lapok (Hungarian Mathematics and Physics Journal for High Schools), Problem A. 321, Vol. 53 (2003/5), p. 297. A Bulgarian Mathematics Competition Problem.

Solution: We are going to do induction on n. The assertion is clearly true in case $n = 1$. Let $n \geq 1$ be fixed, and assume the assertion is true for this n. Let $2(n+1)-1 = 2n+1$ irrational numbers be given. Then, by the assumption, we can select n numbers $x_1, x_2, \ldots, x_n$ among them with the required property. Let the remaining numbers be $x_{n+1}, x_{n+2}, \ldots, x_{2n+1}$, and assume that, for any j with $n+1 \leq j \leq 2n+1$, the numbers $x_1, x_2, \ldots, x_n$, and x_j do not have the required property; that is, we can find nonnegative rational numbers $b_{j1}, b_{j2}, \ldots, b_{jn}$, and b_j not all of which are 0 such that

$$\sum_{i=1}^{n} b_{ji} x_i + b_j x_j = r_j$$

is rational. Here we cannot have $b_j = 0$, since the numbers $x_1, x_2, \ldots, x_n$ satisfy the required property, according to the assumption. That is, with $a_i = b_{ji}/b_j$ and rational $s_j = r_j/b_j$ we have

$$\sum_{i=1}^{n} a_{ji} x_i + x_j = s_j,$$

i.e.,

$$x_j = s_j - \sum_{i=1}^{n} a_{ji} x_i$$

for $j = n+1, n+2, \ldots, 2n+1$.

We claim that then the numbers $x_{n+1}, x_{n+2}, \ldots, x_{2n+1}$ satisfy the required property. To see this, let $c_{n+1}, c_{n+2}, \ldots, c_{2n+1}$ be nonnegative rational numbers not all of which are 0. Then

$$\sum_{j=n+1}^{2n+1} c_j x_j = \sum_{j=n+1}^{2n+1} c_j s_j - \sum_{j=n+1}^{2n+1} c_j \sum_{i=1}^{n} a_{ji} x_i = \sum_{j=n+1}^{2n+1} c_j s_j - \sum_{i=1}^{n} \Big(\sum_{j=n+1}^{2n+1} c_j a_{ji} \Big) x_i.$$

Here the numbers $\sum_{j=n+1}^{2n+1} c_j a_{ji}$ are not zero for all i with $i = 1, 2, \ldots, n$; indeed, if $c_i \neq 0$, then this sum is not zero, either. These sums are rational, so the number

$$\sum_{i=1}^{n} \Big(\sum_{j=n+1}^{2n+1} c_j a_{ji} \Big) x_i$$

must be irrational by our assumption. Since the first sum on the right-hand side is rational, the left-hand side is irrational, as we claimed. The proof is complete.

8) (SENIOR 4) Let $a_1 = 1$, and, for $n \geq 1$, put

$$a_{n+1} = a_n + \frac{1}{\sum_{k=1}^{n} a_k}.$$

Show that the sequence of the numbers a_n is unbounded.

Source: Középiskolai Matematikai és Fizikai Lapok (Hungarian Mathematics and Physics Journal for High Schools), Problem B. 3640, Vol. 53 (2003/4), p. 231.

Solution: It is easy to see that the sequence is increasing. Assume that there is a positive A such that $a_n \leq A$ for all $n \geq 1$. Then,

$$\sum_{k=1}^{n} a_k \leq An,$$

so

$$a_{n+1} \geq a_n + \frac{1}{An}.$$

By induction, it then follows that

$$a_{n+1} \geq \frac{1}{A} \sum_{k=1}^{n} \frac{1}{k}.$$

Indeed, this is true for $n = 1$, since $A \geq a_1 = 1$ and $a_2 = 2$, and the induction step directly follows from the last but one displayed formula. Since the series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

is divergent, the sequence of numbers a_n is not bounded. This is a contradiction, completing the proof.

9) (SENIOR 5) Let n be a positive integer, and let A be an $n \times n$ matrix that satisfies the polynomial equation

$$4A^3 + 3A^2 + 2A + I = O,$$

where I denotes the $n \times n$ identity matrix and O denotes the $n \times n$ zero matrix. Show that A is invertible.

Source: The idea here is well known, but the problem was inspired by Problem 3 on the First Annual Iowa Collegiate Mathematics Competition. See

`http://www.central.edu/maa/Contest/Problems/Probs95.htm`

For more on the Annual Iowa Collegiate Mathematics Competition, see

`http://www.central.edu/maa/Contest/#prevwinners`

Solution: According to the above equation, we have

$$I = -4A^3 - 3A^2 - 2A = A(-4A^2 - 3A - 2I) = (-4A^2 - 3A - 2I)A,$$

showing that the matrix $-4A^2 - 3A - 2I$ is the inverse of A.

10) (SENIOR 6) Find the limit

$$\lim_{n\to\infty} \sum_{k=n+1}^{2n} \frac{1}{k}.$$

Source: Missouri State University Problem Archive, Advanced Problem #10,

`http://faculty.missouristate.edu/l/lesreid/Adv10.html`

For more problems at the site, see

`http://faculty.missouristate.edu/l/lesreid/ADVarchives.html`

Solution: We have

$$\sum_{k=n+1}^{2n} \frac{1}{k} = \sum_{k=n+1}^{2n} \int_{k-1}^{k} \frac{1}{k}\,dx < \sum_{k=n+1}^{2n} \int_{k-1}^{k} \frac{1}{x}\,dx = \int_{n}^{2n} \frac{dx}{x} = \log 2n - \log n = \log 2.$$

Here log denotes the natural logarithm. In calculus courses and in other sciences it is customary to use ln for this; in mathematics, it is more usual to use log for the natural logarithm.

Similarly,

$$\begin{aligned}\sum_{k=n+1}^{2n} \frac{1}{k} &= \sum_{k=n+1}^{2n} \int_{k}^{k+1} \frac{1}{k}\,dx > \sum_{k=n+1}^{2n} \int_{k}^{k+1} \frac{1}{x}\,dx = \int_{n+1}^{2n+1} \frac{dx}{x} = \log(2n+1) - \log(n+1) \\ &= \log\frac{2n+1}{n+1} = \log\left(\frac{2n+2}{n+1} - \frac{1}{n+1}\right) = \log\left(2\left(1 - \frac{1}{2(n+1)}\right)\right) \\ &= \log 2 + \log\left(1 - \frac{1}{2(n+1)}\right).\end{aligned}$$

Now

$$\lim_{n\to\infty} \log\left(1 - \frac{1}{2(n+1)}\right) = \log 1 = 0,$$

where the first equality follows from the continuity of the logarithm. Therefore, the limit in question is $\log 2$.

Remark. We will briefly outline another solution, which uses less elementary tools. We have

$$\begin{aligned}\sum_{k=n+1}^{2n} \frac{1}{k} &= \sum_{k=1}^{2n} \frac{1}{k} - \sum_{k=1}^{n} \frac{1}{k} = \sum_{k=1}^{n} \left(\frac{1}{2k-1} + \frac{1}{2k}\right) - \sum_{k=1}^{n} \frac{1}{k} \\ &= \sum_{k=1}^{n} \left(\frac{1}{2k-1} + \frac{1}{2k} - \frac{1}{k}\right) = \sum_{k=1}^{n} \left(\frac{1}{2k-1} - \frac{1}{2k}\right) = \sum_{k=1}^{2n} (-1)^{k+1} \frac{1}{k}.\end{aligned}$$

The limit of the sum on the right-hand side as n tends to ∞ is

$$\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k}=\log 2,$$

but the proof of this is somewhat subtle; see e.g. Maxwell Rosenlicht, *Introduction to Analysis*, Dover Publications, New York, 1986, p. 155.[4]

11) (SENIOR 7) Let p be an odd prime number, and consider the points in the plane whose coordinates are among the numbers 0, 1, ..., $p-1$. Prove that it is possible to pick p points among these such that no three of the points lie on the same line.

Source: Kürschák József Matematikai Tanulóverseny, 1997 (Student Mathematics Competition in memory of József Kürschák, 1997, Hungary), Problem 1. See

`http://matek.fazekas.hu/portal/feladatbank/egyeb/Kurschak/kurs97/kurs97.html`

See also

`http://matek.fazekas.hu/portal/linkek/index.html`

for the Student Mathematics Portal in Hungary, where there are links to various Hungarian mathematics competitions.

Solution: Pick all pairs of points (k,l) such that $0 \le k < p$ and $l \equiv k^2 \mod p$. Take any three such distinct points (k_1,l_1), (k_2,l_2), (k_3,l_3). To show that these do not lie on the same line, we need to show that the determinant[5]

$$\begin{vmatrix} 1 & 1 & 1 \\ k_1 & k_2 & k_3 \\ l_1 & l_2 & l_3 \end{vmatrix}$$

is not zero. We are going show that this determinant is in fact not congruent to 0 modulo p. The clearest way to do this is to work in the finite field F_p.[6] The elements of this field are the integers 0, 1, ..., $p-1$.[7] One writes $a+b=c$ and $ab=d$ for these integers if one has $a+b\equiv d \mod p$ and $ab\equiv d \mod p$ with the usual operations on integers. The points selected above can be described

[4]The proof in Rosenlicht's book of this equality is based on the Taylor expansion

$$\log(1+x)=\sum_{k=1}^{\infty}(-1)^{k+1}\frac{x^k}{k},$$

valid for x with $-1<x<1$. Then, using the continuity of $\log x$ and the uniform convergence on the interval $[0,1]$ of the series on the right, this equation is extended to $x=1$. The proof above and the present remark can also be viewed as a more direct way of verifying the equality

$$\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k}=\log 2.$$

[5]In fact, this determinant represents twice the area of the triangle in the plane with vertices (k_1,l_1), (k_2,l_2), (k_3,l_3). See

`http://en.wikipedia.org/wiki/Triangle`

[6]See

`http://en.wikipedia.org/wiki/Finite_field`

[7]Instead of these integers, it is more customary to take the elements as the equivalence classes of the integers modulo p, but our choice allows us to simplify the description that follows.

in F_p as the points (k, k^2) for $k \in F_p$. To show that the above determinant is not congruent to 0 modulo p, it is enough to show that we have

$$\begin{vmatrix} 1 & 1 & 1 \\ k_1 & k_2 & k_3 \\ k_1^2 & k_2^2 & k_3^2 \end{vmatrix} \neq 0$$

in F_p for distinct k_1, k_2, and k_3. This determinant is known as the Vandermonde determinant,[8] and its value is well known to be $(k_2 - k_1)(k_3 - k_1)(k_3 - k_2)$, and this is clearly not zero. The proof is complete.

[8] See

http://en.wikipedia.org/wiki/Vandermonde_matrix