

SENIOR PRIZE EXAM  
SPRING 2010

1) Show that there is no integer whose square, written in the decimal system, ends in two odd digits.

2) Let  $n$  be an arbitrary positive integer. Show that

$$n(n+2)(5n-1)(5n+1)$$

is divisible by 24.

3) Show that there is no triangle whose altitudes are of length 4, 7, and 10 units.

4) Let  $a_1 = 1$ , and, for  $n \geq 1$ , put

$$a_{n+1} = a_n + \frac{1}{\sum_{k=1}^n a_k}.$$

Show that the sequence of the numbers  $a_n$  is unbounded.

5) Let  $n$  be a positive integer, and let  $A$  be an  $n \times n$  matrix that satisfies the polynomial equation

$$4A^3 + 3A^2 + 2A + I = O,$$

where  $I$  denotes the  $n \times n$  identity matrix and  $O$  denotes the  $n \times n$  zero matrix. Show that  $A$  is invertible.

6) Find the limit

$$\lim_{n \rightarrow \infty} \sum_{k=n+1}^{2n} \frac{1}{k}.$$

7) Let  $p$  be an odd prime number, and consider the points in the plane whose coordinates are among the numbers  $0, 1, \dots, p-1$ . Prove that it is possible to pick  $p$  points among these such that no three of the points lie on the same line.

SOON AFTER THE EXAM, SOLUTIONS WILL APPEAR ON THE WEB SITE

<http://www.sci.brooklyn.cuny.edu/~mate/prize09/index.html>