All Problems on Prize Exam
Spring 2011
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The source for each problem is listed below when available; but even when the source is given, the formulation of the problem may have been changed. Solutions for the problems presented here were obtained without consulting sources for these solutions even when available, and additional information on how to solve these problems might be obtained by consulting the original sources. There was some overlap between the problems on the Junior and Senior prize exams; the problems common to both exams are listed only once.

1) (JUNIOR 1 and SENIOR 1) Prove that there is no two-digit positive integer that is equal to the product of its digits.

Source: Daniel Arany Mathematical Competition, 9th Grade, Category 1, Round 1, part of Problem 4, Hungary, 1967-68. See

`http://www.versenyvizsga.hu/hun/index.html`

Proof. We need to show that there are no integers x and y with $1 \le x \le 9$, $0 \le y \le 9$, and

$$10x + y = xy.$$

The equation can also be written as

$$10x = (x-1)y.$$

This equation cannot be satisfied by $x = 1$, so we have

$$y = \frac{10x}{x-1} > 10,$$

which is not possible. □

2) (JUNIOR 2 and SENIOR 2) Let n be an integer. Show that $n^2 + 1$ is not divisible by 7.

Source: Daniel Arany Mathematical Competition, 9th Grade, Category 1, Round 1, part of Problem 3, Hungary, 1976-77. See

`http://www.versenyvizsga.hu/hun/index.html`

Proof. The easiest is put the proof in terms of congruences. If $n \equiv 0 \mod 7$ then $n^2+1 \equiv 0^2+1 \equiv 1 \mod 7$. If $n \equiv \pm 1 \mod 7$ then $n^2+1 \equiv (\pm 1)^2 + 1 \equiv 2 \mod 7$. If $n \equiv \pm 2 \mod 7$ then $n^2+1 \equiv (\pm 2)^2 + 1 \equiv 4 \mod 7$. If $n \equiv \pm 3 \mod 7$ then $n^2 + 1 \equiv (\pm 3)^2 + 1 \equiv 10 \equiv 3 \mod 7$. There are no more possibilities. It is clear that in none of these cases does $n^2 + 1 \equiv 0 \mod 7$ holds. The proof is complete. □

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3) (JUNIOR 3 and SENIOR 3) Let x, y, z be positive numbers such that $x + y + z = 1$. Prove that

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} > 3.$$

All computer processing for this manuscript was done under Fedora Linux. The Perl programming language was instrumental in collating the problems. $\mathcal{A}\mathcal{M}\mathcal{S}$-TeX was used for typesetting.

Source: Középiskolai Matematikai és Fizikai Lapok (Mathematical and Physical Journal for Secondary Schools [of Hungary]), September 1908, Problem 1719, p. 7. See

http://db.komal.hu/scan/1908/09/90809007.g4.png

Proof. Given that $x+y+z=1$, we have

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{x+y+z}{x}+\frac{x+y+z}{y}+\frac{x+y+z}{z}$$
$$=1+\frac{y+z}{x}+1+\frac{x+z}{y}+1+\frac{x+y}{z}=3+\frac{y+z}{x}+\frac{x+z}{y}+\frac{x+y}{z}>3;$$

the last inequality holds since x, y, and z are positive. This completes the proof. □

Note. It can be seen by the Cauchy–Schwarz inequality that much more is true. This inequality, stated for real numbers (there is also a version for complex numbers) says that real numbers x_1, x_2, ..., x_n and y_1, y_2, ..., y_n we have

$$\left|\sum_{k=1}^{n} x_k y_k\right|^2 \le \sum_{k=1}^{n} x_k^2 \sum_{k=1}^{n} y_k^2.$$

Taking $x_1=\sqrt{x}$, $x_2=\sqrt{y}$, $x_3=\sqrt{z}$, and $x_k=1/x_k$ here, we obtain that

$$9=(1+1+1)^2=\left(\sqrt{x}\sqrt{\frac{1}{x}}+\sqrt{y}\sqrt{\frac{1}{y}}+\sqrt{z}\sqrt{\frac{1}{z}}\right)^2\le(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)=\frac{1}{x}+\frac{1}{y}+\frac{1}{z}.$$

4) (JUNIOR 4) Given that the product of two of its roots of the equation

$$x^3-14x^2+59x-70=0$$

is 10, solve the equation.

Source: Középiskolai Matematikai és Fizikai Lapok (Mathematical and Physical Journal for Secondary Schools [of Hungary]), April 1911, Problem 2052, p. 198. See

http://db.komal.hu/scan/1911/04/91104198.g4.png

Solution: If x_1, x_2, and x_3 are the solutions of the equation, we have

$$x^3-14x^2+59x-70=(x-x_1)(x-x_2)(x-x_3).$$

Therefore, $x_1x_2x_3=70$. We are given that the product of two of these roots, say x_1 and x_2, is 10. Therefore $x_3=7$. By dividing $x-7$ into $x^3-14x^2+59x-70$, we obtain that

$$x^3-14x^2+59x-70=(x^2-7x+10)(x-7).$$

Now $(x^2-7x+10=(x-2)(x-5)$, so the solutions of the equation are 2, 5, and 7.

5) (JUNIOR 5) One places eighth rooks on a chessboard in such a way that none of the rooks can take any other. Prove that the number of rooks on black squares is even.

Source: Daniel Arany Mathematical Competition, 10th Grade, Category 3, Round 2. Problem 3, Hungary, 2003-04. See

http://www.versenyvizsga.hu/hun/index.html

Solution: A more mathematical way of formulating the problem, for an $n\times n$ chessboard, where n is a positive integer, is as follows:

Lemma. *Let σ be a permutation of the set $S = \{1, 2, 3, \ldots, n\}$; that is, σ is a one-to-one mapping from S onto S. Then the number of integers k with $1 \le k \le n$ such that $k + \sigma(k)$ is odd is even.*

Proof. The number

$$\sum_{k=1}^{n} \bigl(k + \sigma(k)\bigr) = \sum_{k=1}^{n} k + \sum_{k=1}^{n} \sigma(k) = 2 \sum_{k=1}^{n} k$$

is even (the third equality holds since the second sum in the second member of these equations adds the same numbers as the first in a different order). So the number of odd summands in the sum on the left-hand side must be even. □

Note. On a regular chess board the bottom left square is black. In chess, this square is labeled as `a1`, but in a matrix, the corresponding element would usually be called $a_{8,1}$, and $8 + 1$ is odd. On an 8×8 chess board, there the number of rooks on white squares would also be even, but on a 9×9 chess board, the number of k with $k + \sigma(k)$ even would be odd. If we make the top left square white, and we use the usual matrix indexing, squares (i, j) for which $i + j$ is even would be white, and those for which $i + j$ is odd would be black.

6) (JUNIOR 6) Prove that if $n > 2$ is an integer then

$$(2n - 1)^n + (2n)^n < (2n + 1)^n.$$

Source: Daniel Arany Mathematical Competition, Category 2, Round 2, Problem 1, Hungary, 1964-65. See

`http://www.versenyvizsga.hu/hun/index.html`

Proof. If we divide the inequality through by $(2n)^n$, it can be equivalently written as

$$\left(1 + \frac{1}{2n}\right)^n - \left(1 - \frac{1}{2n}\right)^n > 1.$$

To show this, we can use the Binomial Theorem. We obtain

$$\begin{aligned}\left(1 + \frac{1}{2n}\right)^n - \left(1 - \frac{1}{2n}\right)^n &= \sum_{k=0}^{n} \binom{n}{k} \frac{1}{(2n)^k} - \sum_{k=0}^{n} \binom{n}{k} (-1)^n \frac{1}{(2n)^k} \\ &= 2 \sum_{\substack{k: 0 \le k \le n \\ k \text{ is odd}}} \binom{n}{k} \frac{1}{(2n)^k} > 2 \binom{n}{1} \frac{1}{2n} = 1;\end{aligned}$$

note that the last inequality is strict, since we omitted terms in the last sum (since $n > 2$, there is a term corresponding to $k = 3$ in the last sum). This establishes the inequality to be proved. □

7) (JUNIOR 7) Let r and k be integers such that $r \ge 0$ and $0 \le k \le 2^r$. Show that $\binom{2^r - 1}{k}$ is odd.

Solution: The following, more general statement is true.

Let p be a prime and let m, r and k be integers such that $r \ge 0$, $1 \le m \le p$ and $0 \le k \le mp^r - 1$. Then $\binom{mp^r - 1}{k}$ is not divisible by p. (In order to make sense of the case $r = 0$ and $m = 1$, take $\binom{0}{0}$ to be 1).

Proof. To show this, note that

$$\binom{mp^r-1}{k} = \prod_{j=1}^{k} \frac{mp^r-j}{j}.$$

It is clear that for j with $1 \le j \le mp^r - 1$, the numbers j and $mp^r - j$ are divisible by exactly the same powers of p. Therefore, after reducing the fractions $(mp^r - j)/j$ and multiplying them together for j with $1 \le j \le k$, the numerator (and denominator) of the product will not be divisible by p. Since the product is an integer, this integer is not divisible by p, which is what we wanted to show. □

A converse of this statement is also true; see Problem 6 on the Senior Exam.

8) (SENIOR 4) Let n be a positive integer, and let $a_1 < a_2 < a_3 < \dots a_{2n-1}$ real numbers. For which value(s) of the real number x will the function

$$f(x) = \sum_{k=1}^{2n-1} |x - a_k|$$

be minimal.

Source: Daniel Arany Mathematical Competition, 9th Grade, Category 2, Round 2. slightly simplified modification of Problem 3, Hungary, 1969-70. See

`http://www.versenyvizsga.hu/hun/index.html`

Solution: To simplify the discussion, let $a_0 = -\infty$ and $a_{2n} = +\infty$.[1] Let l be an integer with $0 < l \le 2n - 1$ and let x, y be real numbers such that $a_l \le x < y \le a_{l+1}$. It is then easy to see that

$$f(y) - f(x) = l(y - x) - (2n - 1 - l)(y - x) = (2l - 2n + 1)(x - y),$$

since $|a_k - x| = x - a_k$ for $k \le l$ and $|a_k - x| = -x + a_k$ for $k > l$; similarly for y replacing x. This is negative if $l < n$ and it is positive if $l \ge n$. Thus, f is decreasing if $x \le a_n$ and it is increasing if $x \ge a_n$; so f assumes its minimum at a_n.

9) (SENIOR 5) Let f be a function that is continuous in the interval (a, b), and let $c \in (a, b)$. Assume that $f'(x)$ exists for all $x \in (a, b)$ with $x \ne c$, and assume that $\lim_{x\to c} f'(x) = A$. Prove that $f'(c)$ exists and in fact $f'(x) = A$.

Source: Well known, but see W. Rudin, Principles of Mathematical Analysis, 3rd ed., McGraw-Hill, New York, 1976, Problem 9 in Chapter 5, p. 115.

Proof. For $x \in (a, b)$ with $x \ne c$ we have $f(x) - f(c) = f'(\xi_x)(x - c)$ for some ξ_x between x and c (that is, $x < \xi_x < c$ if $x < c$ and $c < \xi_x < x$ if $c < x$) according to the Mean-Value Theorem of Differentiation. Hence

$$f'(c) = \lim_{x\to c} \frac{f(x) - f(c)}{x - c} = \lim_{x\to c} \frac{f'(\xi_x)(x - c)}{x - c} = \lim_{x\to c} f'(\xi_x) = A,$$

which is what we wanted to prove. □

10) (SENIOR 6) Let $n > 0$ be an integer such that that the binomial coefficient $\binom{n}{k}$ is odd for any k with $0 \le k \le n$. Show that $n = 2^r - 1$ for some integer $r > 0$.

Solution: The following statement is also true.

[1] Of course, $-\infty$ and ∞ are not numbers, but they will be only used in a way similar to that in describing an interval such as $(-\infty, +\infty)$.

Let $n \ge 1$ be an integer and let p be a prime. Assume that the binomial coefficient $\binom{n}{k}$ is not divisible by p for any k with $0 \le k \le n$. Then there are integers $r \ge 0$ and m with $1 < m \le p$ such that $n = mp^r - 1$.

Note. Instead of $1 \le m \le p$ we could have required that $1 \le m < p$, since $p \cdot p^r - 1 = 1 \cdot p^{r+1} - 1$. Observe that the case $m = 1$ and $r = 0$ gives $n = 0$; this, however, does not cause any trouble if one takes $\binom{0}{0}$ to be 1; the case $n = 0$ is not interesting, however, and so we assumed $n > 0$. A converse of the above statement is also true; see Problem 7 on the Junior Exam.

Proof. Let $r \ge 0$ be the smallest integer such that $p^{r+1} > n$. Then $n = sp^r + t$ for some integers s and t with $1 \le s < p$ and $0 \le t < p^r$. We need to show that $t = p^r - 1$.

Assume, on the contrary that $t < p^r - 1$; then $0 \le t < p^r - 1$. Note that we have $t + 1 \le n$ since $s \ge 1$. We also have $r > 0$, since for $r = 0$ the inequality $0 \le t < p^r - 1$ cannot be satisfied. Then we have

$$\binom{n}{t+1} = \binom{sp^r + t}{t+1} = \frac{sp^r}{t+1} \prod_{j=1}^{t} \frac{sp^r + j}{j}.$$

As for the first fraction (i.e., the one before the product sign) on the right-hand side, the numerator is divisible by p^r, while the denominator is not. As for the fractions after the product sign, $sp^r + j$ and j for j with $1 \le j \le t$ are divisible by exactly the same power of p. Hence, after reducing these fractions and multiplying them together, we can see that the numerator of the product of all these fractions (including the one before the product sign) is divisible by p while the denominator is not. Since this product is an integer, it is divisible by p, contrary to our assumption that $\binom{n}{k}$ os not divisible by p for any k with $0 \le k \le n$. This completes the proof with $m = s + 1$. □

Comment. One can discuss whether the prime p is a divisor of $\binom{n}{k}$ also in a different way. For any integer $n > 0$, one can determine the exact power of p that divides $n!$ as follows. Let $l \ge 0$ be the integer such that $p^l \mid n!$ but $p^{l+1} \nmid n!$. Then, writing $\lfloor x \rfloor$ for the larger integer m such that $m \le x < m + 1$, called the *integer part*, or *floor*, of x, we have

$$l = \sum_{j=1}^{\infty} \left\lfloor \frac{n}{p^j} \right\rfloor.$$

Note that only finitely many terms on the right-hand side are nonzero, so the sum on the right-hand side is in fact a finite sum. This formula can be explained as follows. Among the numbers 1, 2, $\dots$, n, there are $\lfloor n/p \rfloor$ that are divisible by p. However, there are numbers among these that are divisible by p^2; these need to be counted twice to give l. Hence we add $\lfloor n/p^2 \rfloor$, which counts the numbers divisible by p^2. Those divisible also by p^3 need to be counted three times; hence we also add $\lfloor n/p^3 \rfloor$. By doing this, the numbers divisible by p^3 will indeed be counted three times, since those numbers are also divisible by p and p^2. And so on.

Noting that

$$\binom{n}{k} = \frac{n!}{k!(n-k)!},$$

so if p^l is the exact power of p that divides $\binom{n}{k}$ then we have

$$l = \sum_{j=1}^{\infty} \left(\left\lfloor \frac{n}{p^j} \right\rfloor - \left\lfloor \frac{k}{p^j} \right\rfloor - \left\lfloor \frac{n-k}{p^j} \right\rfloor \right).$$

It is easy to see that $\lfloor x+y \rfloor \geq \lfloor x \rfloor + \lfloor y \rfloor$ for any real numbers x and y. Indeed if s and t are integers such that $s \leq x < s+1$ and $t \leq y < t+1$ then $s+t \leq x+y$, i.e., $\lfloor x \rfloor + \lfloor y \rfloor = s+t \leq \lfloor x+y \rfloor$. Hence we have $l \geq 0$ in the above formula.[2] To ensure that p does not divide $\binom{n}{k}$, we need that

$$\left\lfloor \frac{n}{p^j} \right\rfloor = \left\lfloor \frac{k}{p^j} \right\rfloor + \left\lfloor \frac{n-k}{p^j} \right\rfloor \tag{1}$$

hold for every integer $j \geq 1$, so that the above sum is 0.

Let $j \geq 1$ be an integer such that $p^j \leq n$ and let s and t be integers such that $n = sp^j + t$ and $0 \leq t \leq p^j - 1$; clearly $s \geq 1$. If $t \leq p^j - 2$ then, putting $k = sp^j - 1$ we have $0 \leq n-k = t+1 < p^j$ and so

$$\left\lfloor \frac{n}{p^j} \right\rfloor = s, \qquad \left\lfloor \frac{k}{p^j} \right\rfloor = s-1, \quad \text{and} \quad \left\lfloor \frac{n-k}{p^j} \right\rfloor = 0,$$

and so

$$\left\lfloor \frac{n}{p^j} \right\rfloor - \left\lfloor \frac{k}{p^j} \right\rfloor - \left\lfloor \frac{n-k}{p^j} \right\rfloor = 1,$$

implying that (1) fails for this choice of n and k, i.e., that p is a divisor of $\binom{n}{k}$ in this case. Hence, in order that p not be a divisor of $\binom{n}{k}$ for all k with $0 \leq k \leq n$ we must have $n \equiv -1 \mod p^j$ whenever $p^j \leq n$. Taking r to be the largest integer such that $p^r \leq n$, this holds if and only if $n = mp^r - 1$ for some integer m with $1 < m \leq p$.

Observe that if $n < p$ then (1) holds for every $j \geq 1$ since all three terms in the equation are 0. Note also that in this case $n = mp^0 - 1$ for some m with $1 < m \leq p$, so the above conclusion also holds in case $n < p$.

Conversely, assume that $n = mp^r - 1$ for some integers $r \geq 0$ and m such that $1 < m \leq p$. It is then easy to see that equation (1) holds for all integers $j \geq 1$ and k with $0 \leq k \leq n$. Indeed, let $j \leq r$ and let k be an integer such that $0 \leq k \leq n$. Let s and t be integers such that $k = s2^j + t$ and $0 \leq t \leq 2^j - 1$. Then

$$n = (m2^{r-j} - 1)2^j + (2^j - 1)$$

and

$$n - k = ((m2^{r-j} - 1)2^j + 2^j - 1) - (s2^j + t) = (m2^{r-j} - s - 1)2^j + (2^j - 1 - t).$$

As $0 \leq 2^j - 1 - t < 2^j$, it follows that

$$\left\lfloor \frac{n}{p^j} \right\rfloor = m2^{r-j} - 1, \qquad \left\lfloor \frac{k}{p^j} \right\rfloor = s, \quad \text{and} \quad \left\lfloor \frac{n-k}{p^j} \right\rfloor = m2^{r-j} - s - 1.$$

Hence (1) is satisfied. These considerations establish the following.

Let $n \geq 1$ be an integer and let p be a prime. Then the binomial coefficient $\binom{n}{k}$ is not divisible by p for any k with $0 \leq k \leq n$ if and only if there are integers $r \geq 0$ and m with $1 < m \leq p$ such that $n = mp^r - 1$.

Further comment. The question can also be handled by

[2]This observation can be adapted to show that $\binom{n}{k}$ is an integer – something that we know anyway.

Lucas's Theorem. *Let p be a prime number, and let $m = \sum_{j=0}^{k} m_j p^j$ and $n = \sum_{j=0}^{k} n_j p^j$, where $0 \le m_j < p$ and $0 \le n_j < p$ for j with $0 \le j \le k$ (m_j and n_j are integers). Then*

$$\binom{m}{n} \equiv \prod_{j=0}^{k} \binom{m_j}{n_j} \mod p.$$

Here, for a nonnegative integer r, we can take

$$\binom{x}{r} = \prod_{j=0}^{r-1} \frac{x-j}{r-j}$$

as the definition of the binomial coefficient on the left (if $r = 0$ then the product on the right is the empty product, having value 1). This definition works for any value of x. Observe that if $x < r$ is an integer and $r > 0$ then this definition gives $\binom{x}{r} = 0$, since one of the factors in the numerator is 0 in this case; this is important for understanding the statement of Lucas's theorem, since we may have $m_j < n_j$ there. See

`http://en.wikipedia.org/wiki/Lucas'_theorem`

for a proof of Lucas's theorem, where references to the literature are also given.

11) (SENIOR 7) Let $n \ge 2$ be an integer, and assume that every two rows of an $n \times n$ matrix are different. Prove that it is possible to delete a column such that every two rows of the remaining matrix are still different.

Source: József Kürschák Mathematical Competition, Problem 3, Hungary, 1979/80. See

`http://www.versenyvizsga.hu/hun/index.html`

Solution: In order to use induction to prove this statement, it helps to prove a more general form:

Theorem. *Let m and n be integers with $n \ge m \ge 1$, and assume that every two rows of an $m \times n$ matrix are different. Prove that it is possible to delete a column such that every two rows of the remaining matrix are still different.*

Proof. The case $m = 1$ is vacuously true, since in this case the matrix has only one row. Now, let m and n be integers with $n \ge m \ge 2$, and assume the assertion is true for every $m' \times n$ matrix with $1 \le m' < m$. Let $A = (a(i,j))$ be an $m \times n$ matrix every two rows of which are different. For a k with $1 \le k \le m$ delete the kth row of A. By the induction hypothesis, it is then possible to delete a column, say column $f(k)$, such that every row of the remaining matrix is different. Now, restore row k, but not column $f(k)$. If every two rows of the resulting matrix are different, then we are done, since we deleted a column of the matrix A and every row of the matrix so obtained is different; so assume this is not the case. Then row k agrees with a row $g(k)$ in the resulting matrix, where $g(k) \ne k$. Then we must have $a(k, f(k)) \ne a(g(k), f(k))$, since rows k and $g(k)$ in the matrix A are different.

If we delete a different row k', then we must have $f(k) \ne f(k')$, since if we had $f(k) = f(k')$, then we could delete column $f(k)$ from A and every row of the remaining matrix would be different. That is, f is a one-to-one function.

Now, consider the rows $g^0(1) = k$, $g^1(1) = g(1)$, $g^2(1) = g(g(1))$, $g^3(1) = g(g^2(1))$, etc. Since $1 \le g^r(k) \le m$, there are integers p, q with $0 \le p < q < m$ such that $g^p(1) = g^{q+1}(1)$. If in the matrix A we delete columns $f(g^j(k))$ for $p \le j < q$ then, in the resulting matrix, row $g^j(1)$ agrees with row $g^{j+1}(1)$ for the same values of j, so row $g^p(1)$ agrees with $g^q(1)$ in this matrix. Now, since $g(g^q(1)) = g^p(1)$, we have $a(g^q(1), f(g^q(1)) \neq a(g^p(1), f(g^q(1)))$. However, column $f(g^q(1))$ was not deleted (since f is one-to-one). This is a contradiction, since rows $g^p(1)$ and $g^q(1)$ agree after the deletions. The proof is complete. $\square$