SENIOR PRIZE EXAM
SPRING 2011

1) Prove that there is no two-digit positive integer that is equal to the product of its digits.

2) Let n be an integer. Show that $n^2 + 1$ is not divisible by 7.

3) Let x, y, z be positive numbers such that $x + y + z = 1$. Prove that

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} > 3.$$

4) Let n be a positive integer, and let $a_1 < a_2 < a_3 < \ldots a_{2n-1}$ real numbers. For which value(s) of the real number x will the function

$$f(x) = \sum_{k=1}^{2n-1} |x - a_k|$$

be minimal.

5) Let f be a function that is continuous in the interval (a, b), and let $c \in (a, b)$. Assume that $f'(x)$ exists for all $x \in (a, b)$ with $x \neq c$, and assume that $\lim_{x \to c} f'(x) = A$. Prove that $f'(c)$ exists and in fact $f'(x) = A$.

6) Let $n > 0$ be an integer such that that the binomial coefficient $\binom{n}{k}$ is odd for any k with $0 \leq k \leq n$. Show that $n = 2^r - 1$ for some integer $r > 0$.

7) Let $n \geq 2$ be an integer, and assume that every two rows of an $n \times n$ matrix are different. Prove that it is possible to delete a column such that every two rows of the remaining matrix are still different.

SOON AFTER THE EXAM, SOLUTIONS WILL APPEAR ON THE WEB SITE
`http://www.sci.brooklyn.cuny.edu/~mate/prize11/index.html`

All computer processing for this manuscript was done under Fedora Linux. The Perl programming language was instrumental in collating the problems. $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX was used for typesetting.