All Problems on Prize Exam
Spring 2012
Version Date: Fri Jan 13 19:20:55 EST 2012

The source for each problem is listed below when available; but even when the source is given, the formulation of the problem may have been changed. Solutions for the problems presented here were obtained without consulting sources for these solutions even when available, and additional information on how to solve these problems might be obtained by consulting the original sources. There was some overlap between the problems on the Junior and Senior prize exams; the problems common to both exams are listed only once.

1) (SENIOR 1) Let n be an integer. Prove that

$$n^4 + 6n^3 - n^2 + 18n$$

is divisible by 24.

Source: Exercise 397, Középiskolai Matematikai és Fizikai Lapok Vol. V, No. 5 (1929), p. 157, proposed by László Papp. See

`http://db.komal.hu/scan/`
`http://db.komal.hu/scan/1929/01/92901157.g4.png`

(in Hungarian).

Solution: Writing

$$N = n^4 + 6n^3 - n^2 + 18n,$$

we need to show that N is divisible by 3 and by 8. We have

$$N = (n^4 - n^2) + 3(2n^3 + 6n) = n \cdot (n-1)n(n+1) + 3(2n^3 + 6n).$$

Since one of the numbers $n-1$, n, and $n+1$ is divisible by 3, their product is also divisible by 3. Hence N is divisible by 3.

Furthermore,

$$\begin{aligned} N &= (n^4 - 2n^3 - n^2 + 2n) + 8(n^3 + 2n) = (n^3(n-2) - n(n-2)) + 8(n^3 + 2n) \\ &= (n^3 - n)(n-2) + 8(n^3 + 2n) = (n-2)(n-1)n(n+1) + 8(n^3 + 2n). \end{aligned}$$

There are two even numbers among the numbers $n-2$, $n-1$, n, $n+1$, and one of these is divisible by 4. Thus, their product is divisible by 8, showing that N is also divisible by 8, which is what we wanted to show.

2) (SENIOR 2) Let $n \ge 2$ be an integer, and let $a_1, a_2, \dots, a_n$ be nonzero real numbers, and assume $a_n = a_1$. Show that there is an even number of integers k with $1 \le k \le n-1$ for which $a_k a_{k+1} < 0$.

Solution: Noting that $a_n = a_1$, we have

$$\prod_{k=1}^{n-1} a_k a_{k+1} = \prod_{k=1}^{n-1} a_k^2 > 0;$$

All computer processing for this manuscript was done under Debian Linux. The Perl programming language was instrumental in collating the problems. $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX was used for typesetting.

therefore, there must be an even number of negative factors in the product on the left-hand side.

3) (SENIOR 3) Let α, β, γ be reals such that $\alpha+\beta+\gamma=\pi$. Prove that

$$\sin^2\alpha+\sin^2\beta-\sin^2\gamma=2\sin\alpha\sin\beta\cos\gamma.$$

Source: Problem 454, Középiskolai Matematikai és Fizikai Lapok Vol. V, No. 5 (1929), p. 159. See

`http://db.komal.hu/scan/`
`http://db.komal.hu/scan/1929/01/92901159.g4.png`

(in Hungarian).

Solution: Noting that

$$\sin\gamma=\sin(\pi-\alpha-\beta)=\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta,$$

the left-hand side can be written as

$$\begin{aligned}\sin^2\alpha+\sin^2\beta-\sin^2\gamma&=\sin^2\alpha+\sin^2\beta-(\sin\alpha\cos\beta+\cos\alpha\sin\beta)^2\\&=\sin^2\alpha(1-\cos^2\beta)+\sin^2\beta(1-\cos^2\alpha)-2\sin\alpha\sin\beta\cos\alpha\cos\beta\\&=\sin^2\alpha\sin^2\beta+\sin^2\beta\sin^2\alpha-2\sin\alpha\sin\beta\cos\alpha\cos\beta\\&=2\sin^2\alpha\sin^2\beta-2\sin\alpha\sin\beta\cos\alpha\cos\beta.\end{aligned}$$

Further, we have

$$\cos\gamma=\cos(\pi-\alpha-\beta)=-\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta.$$

Hence, the right-hand side can be written as

$$\begin{aligned}2\sin\alpha\sin\beta\cos\gamma&=-2\sin\alpha\sin\beta(\cos\alpha\cos\beta-\sin\alpha\sin\beta)\\&=2\sin^2\alpha\sin^2\beta-2\sin\alpha\sin\beta\cos\alpha\cos\beta.\end{aligned}$$

Since the right-hand sides we derived are identical, the above identity follows.

4) (SENIOR 4) Let a_1, a_2, $\ldots$ be positive numbers such that

$$\sum_{n=1}^{\infty}a_n<\infty.$$

Prove that there are positive numbers c_n such that

$$\lim_{n\to\infty}c_n=\infty\quad\text{and}\quad\sum_{n=1}^{\infty}c_na_n<\infty.$$

Source: Problem 1, Mathematics Ph. D. Preliminary Exam, Spring 1989, University of California, Berkeley. See

`http://math.berkeley.edu/~desouza/pb.html`
`http://math.berkeley.edu/~desouza/Prelims/Spring89/Spring89.html`

Solution: For any positive integer k, let N_k be the least positive integer such that

$$\sum_{n=N_k}^{\infty} a_n < 2^{-k}.$$

For n with $1 \le n < N_1$, let $c_n = 1$, and for n with $N_k \le n < N_{k+1}$, let $c_n = k$ $(k > 0)$. We have

$$\sum_{n=1}^{\infty} c_n a_n \le \sum_{n=1}^{N_1-1} a_n + \sum_{k=1}^{\infty} k2^{-k} < \infty.$$

5) (SENIOR 5) Find $f(x)$ such that $f''(x) = f(x)f'(x)$, $f(0) = 0$, and $f'(0) = 1/2$. (It is assumed that $f''(x)$ is continuous on the interval where the first equation is satisfied.)

Source: Based on Problem 69 in the Advanced Problem Archive at Missouri State University. See

`http://people.missouristate.edu/lesreid/ADVarchives.html`

Solution: We have

$$f'(x) = \int f(x)f'(x)\,dx.$$

The integral on the right-hand side can be easily evaluated by making the substitution $y = f(x)$. We obtain

$$y' = \int y\,dy = \frac{y^2 + c_1}{2},$$

where c_1 is an arbitrary constant. For $x = 0$ we have $y = 0$ and $y' = 1/2$, which gives $c_1 = 1/2$. That is,

$$\frac{y'}{y^2+1} = \frac{1}{2}.$$

Hence

$$\arctan y = \frac{x + c_2}{2},$$

i.e.,

$$y = \tan\frac{x + c_2}{2}.$$

Here $y = f(x)$. As $f(0) = 0$, we have $c_2 = 0$; that is,

$$f(x) = \tan\frac{x}{2}.$$

6) (SENIOR 6) Let x_n be positive reals such that the series

$$\sum_{n=1}^{\infty} x_n$$

converges, and for each integer $n \ge 1$ let

$$r_n = \sum_{k=n}^{\infty} x_k.$$

Prove that the series

$$\sum_{n=1}^{\infty} \frac{x_n}{r_n}$$

diverges.

Source: Problem 3448 in the American Mathematical Monthly, Vol. 37 (1930), pp. 446-447, proposed by Oliver D. Kellogg. See

`http://www.jstor.org/stable/2298440`

Solution: Assume, on the contrary, that the series $\sum_{n=1}^{\infty} x_n/r_n$ converges, and let N be such that

$$\sum_{n=N}^{\infty} \frac{x_n}{r_n} < 1.$$

Noting that $r_{n+1} = r_n - x_n < r_n$ for every $n \geq 1$, and so $r_n \leq r_N$ for $n \geq N$, we have

$$\sum_{n=N}^{\infty} \frac{x_n}{r_n} \geq \sum_{n=N}^{\infty} \frac{x_n}{r_N} = \frac{1}{r_N} \sum_{n=N}^{\infty} x_n = \frac{1}{r_N} \cdot r_N = 1,$$

which is a contradiction, proving the assertion.

7) (SENIOR 7) Evaluate

$$\int_0^{\pi/2} \ln(\sin x)\, dx.$$

Source: The Harvard-MIT Mathematics Tournament, Calculus Problem 8, February 26, 2000. See

`http://web.mit.edu/hmmt/www/datafiles/problems/`
`http://web.mit.edu/hmmt/www/datafiles/problems/2000.shtml`

Solution: First note that the integral above is an improper integral, since $\lim_{x\searrow 0} \log \sin x = -\infty$. The convergence of the integral is easily established, since $\lim_{x\to\infty} \sin x/x = 1$, and therefore $\lim_{x\searrow 0}(\log \sin x - \log x) = 0$. Below, we will use change of variables in integrals; since these rules are not usually formulated for improper integrals, one needs to check that the applications of these rules are indeed correct, by converting the improper integrals to limits of Riemann integrals.[1] By using the substitution $t = \pi/2 - x$, we can see that

$$\int_0^{\pi/2} \ln \sin x\, dx = -\int_{\pi/2}^{0} \ln \sin\left(\frac{\pi}{2} - t\right)\, dt = \int_0^{\pi/2} \ln \cos t\, dt.$$

Thus,

$$\begin{aligned}
2\int_0^{\pi/2} \ln \sin x\, dx &= \int_0^{\pi/2} (\ln \sin x + \ln \cos x)\, dx = 2\int_0^{\pi/2} \ln(\sin x \cos x)\, dx \\
&= \int_0^{\pi/2} \ln\left(\frac{1}{2}\sin 2x\right)\, dx = \int_0^{\pi/2} \left(\ln\frac{1}{2} + \ln \sin 2x\right)\, dx = \frac{\pi}{2}\ln\frac{1}{2} + \int_0^{\pi/2} \ln \sin 2x\, dx \\
&= \frac{\pi}{2}\ln\frac{1}{2} + \frac{1}{2}\int_0^{\pi} \ln \sin t\, dt = \frac{\pi}{2}\ln\frac{1}{2} + \frac{1}{2}\left(\int_0^{\pi/2} \ln \sin t\, dt + \int_{\pi/2}^{\pi} \ln \sin t\, dt\right);
\end{aligned}$$

[1]If one knows Lebesgue integration theory, then one can observe that the integral is in fact a convergent Lebesgue integral, and the change of variable rule is in fact formulated for Lebesgue integrals.

here the penultimate[2] equality uses the substitution $t = 2x$. On the right-hand side, the two integrals are equal, as can be seen by using the substitution $x = \pi - t$, since $\sin t = \sin(\pi - t)$. Hence we have

$$2\int_0^{\pi/2} \ln \sin x \, dx = \frac{\pi}{2} \ln \frac{1}{2} + \int_0^{\pi/2} \ln \sin t \, dt,$$

and so

$$\int_0^{\pi/2} \ln \sin x \, dx = \frac{\pi}{2} \ln \frac{1}{2} = -\frac{\pi}{2} \ln 2.$$

[2]The one before the last one.