Senior Prize Exam
Spring 2012

1) Let n be an integer. Prove that

$$n^4 + 6n^3 - n^2 + 18n$$

is divisible by 24.

2) Let $n \geq 2$ be an integer, and let a_1, a_2, ..., a_n be nonzero real numbers, and assume $a_n = a_1$. Show that there is an even number of integers k with $1 \leq k \leq n-1$ for which $a_k a_{k+1} < 0$.

3) Let α, β, γ be reals such that $\alpha + \beta + \gamma = \pi$. Prove that

$$\sin^2\alpha + \sin^2\beta - \sin^2\gamma = 2\sin\alpha\sin\beta\cos\gamma.$$

4) Let a_1, a_2, ... be positive numbers such that

$$\sum_{n=1}^{\infty} a_n < \infty.$$

Prove that there are positive numbers c_n such that

$$\lim_{n\to\infty} c_n = \infty \quad \text{and} \quad \sum_{n=1}^{\infty} c_n a_n < \infty.$$

5) Find $f(x)$ such that $f''(x) = f(x)f'(x)$, $f(0) = 0$, and $f'(0) = 1/2$. (It is assumed that $f''(x)$ is continuous on the interval where the first equation is satisfied.)

6) Let x_n be positive reals such that the series

$$\sum_{n=1}^{\infty} x_n$$

converges, and for each integer $n \geq 1$ let

$$r_n = \sum_{k=n}^{\infty} x_k.$$

Prove that the series

$$\sum_{n=1}^{\infty} \frac{x_n}{r_n}$$

diverges.

7) Evaluate

$$\int_0^{\pi/2} \ln(\sin x)\,dx.$$

Soon after the exam, solutions will appear on the Web Site
`http://www.sci.brooklyn.cuny.edu/~mate/prize12/index.html`

All computer processing for this manuscript was done under Debian Linux. The Perl programming language was instrumental in collating the problems. $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX was used for typesetting.