All Problems on Prize Exam
Spring 2013
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The source for each problem is listed below when available; but even when the source is given, the formulation of the problem may have been changed. Solutions for the problems presented here were obtained without consulting sources for these solutions even when available, and additional information on how to solve these problems might be obtained by consulting the original sources. There was some overlap between the problems on the Junior and Senior prize exams; the problems common to both exams are listed only once.

1) (SENIOR 1) Prove that the product of two consecutive positive integers is not a square of an integer.

Source: Problem 202, Középiskolai Matematikai és Fizikai Lapok Vol. III, No. 3 (1926), p. 92,
`http://db.komal.hu/scan/`
`http://db.komal.hu/scan/1926/11/92611092.g4.png`
(in Hungarian).

Solution: Given a positive integer n, its square is n^2, and the square of the next integer is $(n+1)^2 = n^2 + 2n + 1$. Since $n(n+1) = n^2 + n$ is between these two numbers, it cannot be the square of an integer.

2) (SENIOR 2) Show that $\sqrt{2}$, $\sqrt{5}$, and $\sqrt{7}$ cannot belong to the same geometric progression.

Source: Problem 205, Középiskolai Matematikai és Fizikai Lapok Vol. III, No. 3 (1926), p. 92,
`http://db.komal.hu/scan/`
`http://db.komal.hu/scan/1926/11/92611092.g4.png`
(in Hungarian).

Solution: Assuming that these numbers belong to the same geometric progression $a_n = aq^n$, $n = 0, 1, 2, \ldots$, where $a > 0$, $q > 0$, and $q \ne 1$, we have $\sqrt{2} = aq^{k_1}$, $\sqrt{5} = aq^{k_2}$, $\sqrt{7} = aq^{k_3}$ for some nonnegative integers k_1, k_2, and k_3. Then $\sqrt{5/2} = q^{k_2-k_1}$ and $\sqrt{7/2} = q^{k_3-k_1}$, and so

$$\frac{7}{2} = \left(\frac{5}{2}\right)^{k/l}$$

with $k = k_3 - k_1$ and $l = k_2 - k_1$; note that, clearly, $k_1 < k_2 < k_3$, and so $k > l > 0$, in case $q > 1$, and $k_1 > k_2 > k_3$, and so $k < l < 0$, in case $0 < q < 1$. and $l > 0$. Thus, we have

$$7^l \cdot 2^{k-l} = 5^k.$$

This is impossible in case $k > l > 0$, because no positive integer power of 5 is divisible by 2. In case $k < l < 0$, write this equation equivalently as

$$7^{-l} \cdot 2^{-(k-l)} = 5^{-k},$$

so that all the exponents are positive. This equation is impossible for the same reason.

3) (SENIOR 3) Prove that an integer n is the sum of the squares of two integers if and only if $2n$ has the same property (i.e., $2n$ is also the sum of the squares of two integers; note that one or both of these integers may or may not be zero).

All computer processing for this manuscript was done under Debian Linux. The Perl programming language was instrumental in collating the problems. $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX was used for typesetting.

Source: Középiskolai Matematikai és Fizikai Lapok Vol. XV, No. 3 (1938), Problem 1461, p. 80,

http://db.komal.hu/scan/

http://db.komal.hu/scan/1938/11/93811080.g4.png

(in Hungarian).

Solution: If $n^2 = x^2 + y^2$ for some integers x and y then $2n = 2x^2 + 2y^2 = (x+y)^2 + (x-y)^2$. This shows that if n can be represented as the sum of squares of two integers then $2n$ can also be so represented.

If, on the other hand, $2n = u^2 + v^2$ for some integers u and v then $n = x^2 + y^2$ with x and y such that $u = x + y$ and $v = x - y$, that is, with

$$x = \frac{u+v}{2} \quad \text{and} \quad y = \frac{u-v}{2}.$$

Observing the equation $2n = u^2 + v^2$ implies that u and v have the same parity (i.e., that either they are both even or they are both odd), it follows that x and y are integers. Hence it also follows that if $2n$ can be represented as the sum squares of two integers then n can also be so represented.

4) (SENIOR 4) Evaluate

$$\sum_{n=0}^{\infty} \frac{1}{(2n)!}.$$

Source: A special case of Problem 77 in the Advanced Problem Archive at Missouri State University. See

http://people.missouristate.edu/lesreid/ADVarchives.html

Solution: We have

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Hence

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} \quad \text{and} \quad \frac{1}{e} = e^{-1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}.$$

Therefore

$$\frac{e + 1/e}{2} = \sum_{n=0}^{\infty} \frac{1 + (-1)^n}{2 \cdot n!} = \sum_{n=0}^{\infty} \frac{1}{(2n)!};$$

the second equation holds since

$$\frac{1 + (-1)^n}{2} = \begin{cases} 1 & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

5) (SENIOR 5) Given positive real numbers a_n such that $a_n < a_{n+1} + a_n^2$, prove that $\sum_{n=1}^{\infty} a_n$ is divergent.

Source: Problem P. 398, Középiskolai Matematikai Lapok Vol. 34, No. 5 (1984), p. 222,

http://db.komal.hu/scan/

http://db.komal.hu/scan/1984/05/98405222.g4.png

(in Hungarian).

Solution: Assume $\sum_{n=1}^{\infty} a_n$ is convergent. Then $\lim_{n\to\infty} a_n = 0$. Pick an integer $n \ge 1$ such that

$$\sum_{k=n}^{\infty} a_k < 1$$

and $a_n \ge a_k$ for all integers $k > n$. Such an n can be found as follows. Let $n_1 \ge 1$ be such that $\sum_{k=n_1}^{\infty} a_k < 1$, and pick $n_2 > n_1$ such that $a_{n_2} > a_{n_1}$ if possible, and then pick $n_3 > n_2$ such that $a_{n_3} > a_{n_2}$ if possible, and so on. Now, there cannot be an infinite sequence $\{a_{n_k}\}_{k=1}^{\infty}$ such that $a_{n_i} > a_{n_j}$ whenever $1 \le i < j$ since $\lim_{n\to\infty} a_n = 0$. So let $N \ge 1$ be such that we have $a_{n_i} > a_{n_j}$ whenever $1 \le i < j \le N$, and assume that this sequence cannot be continued, i.e., that there is no $k > n_N$ such that $a_k > a_{n_N}$. Then $n = n_N$ will have the desired property, i.e., we will have $a_n \ge a_k$ for $k > n$. We have $a_k < a_{k+1} + a_k^2$ for all $k \ge 1$ according to the assumptions. Taking an arbitrary $m > n$ and summing this for k with $n \le k \le m$, we obtain that

$$\sum_{i=n}^{m} a_k < \sum_{i=n}^{m} a_{k+1} + \sum_{i=n}^{m} a_k^2,$$

i.e., that

$$a_n < a_{m+1} + \sum_{i=n}^{m} a_k^2 \le a_{m+1} + \sum_{i=n}^{m} a_n a_k = a_{m+1} + a_n \sum_{i=n}^{m} a_k,$$

where the second inequality follows by noting that $a_k^2 = a_k a_k \le a_n a_k$ for $k > n$, since $a_k \le a_n$ for $k > n$ according to the choice of n. Making $m \to \infty$ and noting that $a_{m+1} \to 0$, this implies that

$$a_n \le a_n \sum_{i=n}^{\infty} a_k.$$

This is a contradiction, since $\sum_{k=n}^{\infty} a_k < 1$ according to the choice of n, showing that $\sum_{n=1}^{\infty} a_n$ is divergent.

6) (SENIOR 6) Assume f is continuous on $[0, +\infty)$, differentiable on $(0, +\infty)$, f' is strictly decreasing on $(0, +\infty)$, and $f(0) = 0$. Prove that $f(x)/x$ is strictly decreasing on $(0, +\infty)$.

Source: Problem 5, Ohio State University Mathematics Ph. D. Qualifying Examination in Analysis, Autumn 2003. See

`http://www.math.osu.edu/graduate/current/qual`

Solution: Let x, y be real numbers such that $0 < x < y$. By the Mean-Value Theorem of Differentiation, we have

$$f(x) = f(x) - f(0) = xf'(\xi) \tag{1}$$

for some ξ with $0 < \xi < x$, and

$$f(y) - f(x) = (y - x)f'(\eta)$$

for some η with $x < \eta < y$. Since $\xi < \eta$, we have $f'(\xi) > f'(\eta)$. Hence

$$\begin{aligned} f(y) &= (y - x)f'(\eta) + f(x) < (y - x)f'(\xi) + f(x) \\ &= (y - x)f'(\xi) + xf'(\xi) = yf'(\xi) = \frac{y}{x}\, xf'(\xi) = \frac{y}{x} f(x). \end{aligned}$$

Thus

$$\frac{f(y)}{y} < \frac{f(x)}{x},$$

which is what we wanted to prove.

Note: In a somewhat different proof, one may note that $f'(x) < f'(\xi)$ in equation (1), and so

$$f(x) > xf'(x)$$

for $x > 0$. Hence

$$\left(\frac{f(x)}{x}\right)' = \frac{xf'(x) - f(x)}{x^2} < 0$$

for all $x > 0$, showing that $f(x)/x$ is strictly decreasing.

7) (SENIOR 7) Let f be a continuous real-valued function in the interval $[0,1]$ satisfying the inequality $xf(y) + yf(x) \le 1$ for any $x, y \in [0,1]$. Show that $\int_0^1 f(x)\,dx \le \pi/4$.

Source: The International Mathematics Competition, 1998, Problem 6. Follow the link at the site

`http://www.artofproblemsolving.com/Forum/resources.php`

to IMC (under the heading "Undergraduate Competitions," in the middle of the page).

Solution: Let $t \in [0, \pi/2]$. Then, with $x = \sin t$ and $y = \cos t$ we have

$$f(\cos t)\sin t + f(\sin t)\cos t \le 1.$$

Integrating this on $[0, \pi/2]$, we obtain that

$$\int_0^{\pi/2} f(\cos t)\sin t\,dt + \int_0^{\pi/2} f(\sin t)\cos t\,dt \le \int_0^{\pi/2} dt = \frac{\pi}{2}.$$

Noting that the substitutions $x = \cos t$ and $x = \sin t$ give, respectively, that

$$\int_0^1 f(x)\,dx = -\int_{\pi/2}^0 f(\cos t)\sin t\,dt = \int_0^{\pi/2} f(\cos t)\sin t\,dt,$$

and

$$\int_0^1 f(x)\,dx = \int_0^{\pi/2} f(\sin t)\cos t\,dt,$$

the left-hand side of the above inequality equals $2\int_0^1 f(x)\,dx$, and so the inequality

$$\int_0^1 f(x)\,dx \le \frac{\pi}{4}.$$

follows.

Note that the function $f(x) = \sqrt{1-x^2}$ satisfies the assumptions, since, for any x, y in the interval $[0,1]$ we have

$$\begin{aligned} x\sqrt{1-y^2} + y\sqrt{1-x^2} &\le \sqrt{y^2+x^2}\sqrt{(1-x^2)+(1-y^2)} \\ &\le \frac{(y^2+x^2) + ((1-x^2)+(1-y^2))}{2} = 1, \end{aligned}$$

where the first inequality follows from the Cauchy–Schwarz inequality, and the second inequality holds by the inequality between the geometric and arithmetic means. Furthermore, for this choice of f we have

$$\begin{aligned}\int_0^1 f(x)\,dx &= \int_0^1 \sqrt{1-x^2}\,dx = \int_0^{\pi/2} \cos^2 t\,dt = -\int_{\pi/2}^0 \sin^2 t\,dt \\ &= \int_0^{\pi/2} \sin^2 t\,dt = \frac{1}{2}\int_0^{\pi/2} (\sin^2 t + \cos^2 t)\,dt = \frac{1}{2}\int_0^{\pi/2} dt = \frac{\pi}{4},\end{aligned}$$

where the second equality is based on the substitution $x = \sin t$ and the third equality, on the substitution $x = \cos t$; the fifth equality follows by taking the arithmetic mean of the third and the fifth members of these equalities. That is, the inequality $\int_0^1 f(x)\,dx \le \pi/4$ cannot be improved.