Senior Prize Exam
Spring 2013

1) Prove that the product of two consecutive positive integers is not a square of an integer.

2) Show that $\sqrt{2}$, $\sqrt{5}$, and $\sqrt{7}$ cannot belong to the same geometric progression.

3) Prove that an integer n is the sum of the squares of two integers if and only if $2n$ has the same property (i.e., $2n$ is also the sum of the squares of two integers; note that one or both of these integers may or may not be zero).

4) Evaluate

$$\sum_{n=0}^{\infty} \frac{1}{(2n)!}.$$

5) Given positive real numbers a_n such that $a_n < a_{n+1} + a_n^2$, prove that $\sum_{n=1}^{\infty} a_n$ is divergent.

6) Assume f is continuous on $[0, +\infty)$, differentiable on $(0, +\infty)$, f' is strictly decreasing on $(0, +\infty)$, and $f(0) = 0$. Prove that $f(x)/x$ is strictly decreasing on $(0, +\infty)$.

7) Let f be a continuous real-valued function in the interval $[0, 1]$ satisfying the inequality $xf(y) + yf(x) \le 1$ for any $x, y \in [0, 1]$. Show that $\int_0^1 f(x)\,dx \le \pi/4$.

Soon after the exam, solutions will appear on the Web Site
`http://www.sci.brooklyn.cuny.edu/~mate/prize12/index.html`

All computer processing for this manuscript was done under Debian Linux. The Perl programming language was instrumental in collating the problems. $\mathcal{A}\mathcal{M}\mathcal{S}$-TeX was used for typesetting.