All Problems on Prize Exam
Spring 2014

The source for each problem is listed below when available; but even when the source is given, the formulation of the problem may have been changed. Solutions for the problems presented here were obtained without consulting sources for these solutions even when available, and additional information on how to solve these problems might be obtained by consulting the original sources. There was some overlap between the problems on the Junior and Senior prize exams; the problems common to both exams are listed only once.

1) (JUNIOR 1 and SENIOR 1) Given a positive integer n, show that n^3+5n is divisible by 6.

Solution: The assertion is true for any integer n, positive or negative. Indeed, we have

$$n^3+5n=(n^3-n)+6n=(n-1)n(n+1)+6n.$$

Among the three consecutive integers $n-1$, n, and $n+1$, there must be one that is divisible by 2, and also one that is divisible by 3.

2) (JUNIOR 2 and SENIOR 2) Show that there are no four consecutive integers (i.e., integers of form n, $n+1$, $n+2$, $n+3$ for some n) each of which is a power with an integer exponent >1 of an integer.

Source: Problem 947, Középiskolai Matematikai Lapok Vol. XVIII, No. 1 (1959), p. 29, proposed by Paul Erdős; see

`http://db.komal.hu/scan/1959/01/95901029.g4.png`

Solution: Among the four consecutive integers, there will be one that is divisible by 2 and not divisible by 4. That number cannot be a power of another integer with an exponent >1.

3) (JUNIOR 3 and SENIOR 3) Let a, b, c, d, p, and q be positive integers satisfying $ad-bc=1$ and $a/b>p/q>c/d$. Prove that $q\ge b+d$.[1]

Source: Problem 2 on the IberoAmerican International Mathematical Competition, 1988. See

`http://www.artofproblemsolving.com/Forum/resources.php`

Solution: We have $cq<pd$. Since both sides are integers, we in fact have $cq+1\le pd$, and so

$$bcq+b\le pdb.$$

Similarly, $pb<aq$, and therefore $pb+1\le aq$; hence

$$pbd+d\le adq.$$

Combining the two displayed inequalities, we obtain

$$bcq+b+d\le adq.$$

Finally, noting that $ad=bc+1$, this implies that

$$bcq+b+d\le bcq+q,$$

All computer processing for this manuscript was done under Debian Linux. The Perl programming language was instrumental in collating the problems. $\mathcal{A}\mathcal{M}\mathcal{S}$-TeX was used for typesetting.

[1] By mistake, the our earlier formulation of the problem omitted d from the list of positive integers, and the text said: Let a, b, c, p, and q be positive integers satisfying $ad-bc=1$ and $a/b>p/q>c/d$. Prove that $q\ge b+d$.

whence $q \geq b + d$ follows.

4) (JUNIOR 4) Let $n \geq 0$ be an integer. Show that $3^n + 1$ is not divisible by 8.

Solution: We have $3^2 \equiv 1 \mod 8$; raising this the the power k, where $k \geq 0$ is an integer, we obtain $3^{2k} \equiv 1 \mod 8$. Multiplying this by 3, we can see that $3^{2k+1} \equiv 3 \mod 8$. Hence $3^n + 1 \equiv 4 \mod 8$ if n is even, and $3^n + 1 \equiv 2 \mod 8$ if n is odd.

5) (JUNIOR 5) Let n and k be positive integers. Find the number of k element subsets of the set $\{1, 2, \dots, n\}$ that contain no consecutive integers (two integers are called consecutive if there difference is 1).

Source: Based on Problem 4(a), Mathematics Qualifying Examination in Combinatorics, Arizona State University, December 2011; see

`http://math.asu.edu/degree-programs/past-qualifying-examinations`

Solution: Let $A = \{a_0, a_1, \dots, a_{k-1}\}$ be a k-element subset of $\{1, 2, \dots, n\}$ such that no two elements of it are consecutive, with its elements listed in increasing order. Then the set $f(A) = \{a_i - i : 0 \leq i \leq k-1\}$ is a subset of $\{1, 2, \dots, n-k+1\}$. The mapping f is a one-to-one mapping of the set of all k-element subsets of $\{1, 2, \dots, n\}$ with no consecutive elements onto the set of all k-element subsets of $\{1, 2, \dots, n-k+1\}$. The number of elements of the latter set is $\binom{n-k+1}{k}$; in case $k > n-k+1$, this binomial coefficient is taken to be 0. This is also the number of elements of the former set.

6) (JUNIOR 6) Given an integer $n > 0$, show that

$$\sum_{k=0}^{n} \binom{2n}{2k} 3^k$$

is divisible by 2^n Problem B. 3489, Középiskolai Matematikai Lapok 2001/7, p. 424,

Source:

`http://db.komal.hu/scan/2001/10/MAT0107.PS.png.38`

Solution: We have

$$\sum_{k=0}^{n} \binom{2n}{2k} 3^k = \sum_{l=0}^{2n} \binom{2n}{l} (\sqrt{3})^l + \sum_{l=0}^{2n} \binom{2n}{l} (\sqrt{3})^l (-1)^l.$$

Noting that $(-1)^l = (-1)^{2n-l}$, according to the Binomial Theorem the right-hand side equals

$$\begin{aligned}(\sqrt{3}+1)^{2n} + (\sqrt{3}-1)^{2n} &= ((\sqrt{3}+1)^2)^n + ((\sqrt{3}-1)^2)^n \\ &= (4+2\sqrt{3})^2 + (4-2\sqrt{3})^n = 2^n\big((2+\sqrt{3})^n + (2-\sqrt{3})^n\big).\end{aligned}$$

Another application of the binomial theorem shows that the expression multiplied by 2^n on the right-hand side is an integer:

$$\begin{aligned}(2+\sqrt{3})^n + (2-\sqrt{3})^n &= \sum_{l=0}^{n} \binom{n}{l} 2^{n-l} (\sqrt{3})^l + \sum_{l=0}^{n} \binom{n}{l} 2^{n-l} (-\sqrt{3})^l \\ &= \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} 2^{n-2k} 3^k.\end{aligned}$$

Hence the number above is indeed divisible by 2^n.

7) (JUNIOR 7) Assume that for a triangle with angles α, β, and γ, we have

$$\sin\gamma = \cos\alpha + \cos\beta.$$

Show that α or β must be a right angle.

Source: Problem 1467, Középiskolai Matematikai és Fizikai Lapok Vol. XV, No. 3 (1938), Problem 1467, p. 81,

`http://db.komal.hu/scan/`
`http://db.komal.hu/scan/1938/11/93811081.g4.png`

(in Hungarian).

Solution: We have $\alpha+\beta+\gamma=\pi$, so $\gamma=\pi-\alpha=\beta$. That is, we have

$$\sin(\pi-\alpha-\beta) = \cos\alpha + \cos\beta.$$

As $\sin(\pi-\alpha-\beta)=\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$, we have

$$\sin\alpha\cos\beta+\cos\alpha\sin\beta = \cos\alpha+\cos\beta,$$

i.e.,

$$-\cos\beta(1-\sin\alpha) = \cos\alpha(1-\sin\beta).$$

This is certainly true if $\alpha=\pi/2$ or if $\beta=\pi/2$, since both sides are zero in either of these cases.

Assuming that this is not the case, and given that $0<\alpha<\pi$ and $0<\beta<\pi$ (since α and β are angles of a triangle), we will show that this equality cannot hold. In fact, assuming it holds, we have

$$\frac{\cos\alpha}{1-\sin\alpha} = -\frac{\cos\beta}{1-\sin\beta},$$

that is,

$$\frac{\cos\alpha}{1-\sin\alpha} = \frac{\cos(\pi-\beta)}{1-\sin(\pi-\beta)},$$

Since we have $\alpha+\beta<\pi$, $\alpha>0$, and $\beta>0$, we may assume here that $0<\alpha<\pi/2$; namely, all our equations are symmetric in α and β. The left-hand side is positive in this case; in order for the right-hand side also to be positive, we must have $0<\pi-\beta<\pi/2$. Now, the function

$$f(x)=\frac{\cos x}{1-\sin x}$$

in increasing in the interval $(0,\pi/2)$, as we will show below, so the above equation can hold only in case $\alpha=\pi-\beta$, which is of course impossible, since $\alpha+\beta=\pi-\gamma<\pi$.

To show that f increasing in the interval $(0,\pi/2)$, one may observe that

$$\begin{aligned} f'(x) &= \frac{-\sin x(1-\sin x)-\cos x(-\cos x)}{(1-\sin x)^2} = \frac{\cos^2 x+\sin^2 x-\sin x}{(1-\sin x)^2} \\ &= \frac{1-\sin x}{(1-\sin x)^2} = \frac{1}{1-\sin x} > 0 \qquad \text{if} \quad 0<x<\frac{\pi}{2}. \end{aligned}$$

Another way to show that f increasing in the interval $(0, \pi/2)$ proceeds by noting that

$$f(t-\pi/2) = \frac{\cos(t-\pi/2)}{1-\sin(t-\pi/2)} = \frac{\cos(\pi/2-t)}{1+\sin(\pi/2-t)} = \frac{\sin t}{1+\cos t} = \tan\frac{t}{2}.$$

Hence, putting $t = x + \pi/2$ we obtain

$$f(x) = \tan\frac{x+\pi/2}{2}.$$

The right-hand side is clearly increasing in any interval where it is is continuous, showing that f is indeed increasing in the interval $(0, \pi/2)$.

8) (SENIOR 4) Show that

$$\int_{-1}^{1} \ln(x+\sqrt{1+x^2})\,dx = 0.$$

Source: Simplified version of a problem on the final of the Estonian Mathematical Olympiad, 1999. See Problem A29.5 at

`http://matek.fazekas.hu/portal/feladatbank/gyujtemenyek/Nem/AF29.htm`

Solution: Writing $f(x)$ for the integrand, it is easy to show that the integrand is an odd function, that is $f(-x) = -f(x)$. Indeed, we have

$$(x+\sqrt{1+x^2})(-x+\sqrt{1+(-x)^2}) = (1+x^2) - x^2 = 1,$$

and so $f(x) + f(-x) = 0$. Hence $\int_{-1}^{1} f(x)\,dx = 0$.

Note. One may also observe that the indefinite integral

$$\int \ln(x+\sqrt{1+x^2})\,dx$$

is not difficult to calculate. Using integration by parts, we have

$$\begin{aligned}\int \ln(x+\sqrt{1+x^2})\,dx &= \int 1\cdot\ln(x+\sqrt{1+x^2})\,dx \\ &= x\ln(x+\sqrt{1+x^2}) - \int x\frac{1+x/\sqrt{1+x^2}}{x+\sqrt{1+x^2}}\,dx = x\ln(x+\sqrt{1+x^2}) - \int \frac{x}{\sqrt{1+x^2}}\,dx\end{aligned}$$

Using the substitution $t = \sqrt{1+x^2}$, $dt = 2x, dx$, we obtain

$$\int \frac{x}{\sqrt{1+x^2}}\,dx = \int \frac{1}{2\sqrt{t}}\,dt = \sqrt{t} + C = \sqrt{1+x^2} + C.$$

Hence

$$\int \ln(x+\sqrt{1+x^2})\,dx = x\ln(x+\sqrt{1+x^2}) - \sqrt{1+x^2} + C.$$

One can use this result to obtain the solution of the problem.

9) (SENIOR 5) For a given integer $n \geq 2$, place n red points and n blue points in a row. A place between two adjacent points will be called an *even split* if cutting the row at that place will leave the same number of red and blue points to the left of the cut. Show that the number of color arrangements with exactly one even split is twice the number of color arrangements with no even split.

Source: Problem 1, József Kürschák Mathematical Competition, Hungary, 1972. See

`http://matek.fazekas.hu/list.php?what=competition`

(in Hungarian).

Solution: In counting the color arrangements, individual points of the same color are not distinguished. In the original formulation of the problem, boys and girls in a class room are mentioned, where in counting the arrangements, it is natural to distinguish between persons. This, however, does not affect the question being asked; each color arrangement corresponds to $(n!)^2$ person arrangements, where the girls and boys are permuted among themselves.

In solving problem, yet a third interpretation will prove useful. Consider walks consisting of $2n$ moves in the coordinate plane as follows. A walk starts out at the point $(0,0)$, it ends at the point $(2n,0)$, and each move takes one from a point (k,l) either to the point $(k+1,l+1)$, called an *up* move, or to the point $(k+1,l-1)$, called a *down* move. We need to show that the number of walks that meet the x axis between its starting and ending points exactly once is twice the number of walks that do not meet the x axis.

In counting these walks, let A_n be the number of walks that always stay above the x axis (except at the starting and ending points), let B_n be the number of those that never go below the x axis, and let C_n be the number of those that meet the x axis exactly once, and never goes below the x axis.

The number of those walks that never meet the x axis is $2A_n$ (those walks that always stay above the x axis, and their reflections with respect to the x axis), and those that meet the x axis exactly once is $4C_n$; this is so, because for a walk that meets the x axis exactly once but otherwise stays above the x axis, we can independently decide to reflect or not to reflect the part before the meeting point, and after the meeting point, with respect to the x axis. Thus, we need to show that $A_n = C_n$ for $n \geq 2$.

In counting these walks, we can relax the restriction $n \geq 2$, and allow any $n \geq 0$. We have $A_0 = B_0 = 1$ and $C_0 = 0$. Further, we have

$$A_n = B_{n-1} \qquad (n \geq 1),$$

because a walk that stays always above the x axis must start with an up move, end with a down move, and the part of the walk between the points $(1,1)$ and $(2n-1,1)$ never goes below the line $x = 1$. Next, we have

$$B_n = \sum_{k=1}^{n} A_k B_{n-k} \qquad (n \geq 1).$$

Here, the term $A_k B_{n-k}$ represents those walks that first touch the x axis at the point $(2k,0)$; before this point, it stays above the x axis, and never goes below the x axis after this point. For this, it is important that we put $B_0 = 1$, which is needed in case $k = n$. Finally,

$$C_n = \sum_{k=1}^{n-1} A_k A_{n-k} \qquad (n \geq 1).$$

Here, the term $A_k A_{n-k}$ represents those walks that touch the x axis at the point $(2k, 0)$, and stays above the x axis before and after this point. While this last calculation does not make much sense in case $n = 1$, it correctly gives the result $C_1 = 0$, the sum being the empty sum in this case.

Hence, using these displayed equations, for $n \geq 2$ we have

$$A_n = B_{n-1} = \sum_{k=1}^{n-1} A_k B_{n-1-k} = \sum_{k=1}^{n-1} A_k A_{n-k} = C_n,$$

establishing the equality $A_n = C_n$.

Note: Using the above formulas, it is possible to obtain an expression for B_n. According to the first and second displayed formulas above, we have

$$B_n = \sum_{k=1}^{n} B_{k-1} B_{n-k} = B_n = \sum_{j=0}^{n-1} B_j B_{n-1-j} \qquad (n \geq 1),$$

where, to obtain the second equality, we wrote $j = k - 1$. Writing

$$f(x) = \sum_{n=0}^{\infty} B_n x^n,$$

and assuming that the series on the right-hand side has a positive radius of convergence, noting that $B_0 = 1$, by the recursive formula above we obtain

$$f(x) = 1 + x \sum_{m=1}^{\infty} x^m B_{m+1} = 1 + x \sum_{m=1}^{\infty} x^m \sum_{j=0}^{m} B_j B_{m-j} = 1 + x(f(x))^2$$

inside the interval of convergence of the series of $f(x)$; the function f is called the *generating function* of the sequence of the numbers B_n. Thus,

$$x(f(x))^2 - f(x) + 1 = 0.$$

Solving this equation for $f(x)$, we obtain

$$f(x) = \frac{1 \pm \sqrt{1 - 4x}}{2x}.$$

Noting that $f(x)$ is bounded in a neighborhood of $x = 0$, we must have the $-$ sign in the $\pm$ in this formula. Using the binomial series

$$(1 + t)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n \qquad (|x| < 1),$$

where

$$\binom{\alpha}{n} = \prod_{k=0}^{n-1} \frac{\alpha - k}{k + 1} = \frac{1}{n!} \prod_{k=0}^{n-1} (\alpha - k) \qquad (n \geq 0),$$

for $|4x| < 1$ we obtain

$$f(x) = \frac{1}{2x}(1-(1-4x)^{1/2}) = \frac{1}{2x}\left(1-\sum_{n=0}^{\infty}\binom{1/2}{n}(-4x)^n\right) = \sum_{n=1}^{\infty}\binom{1/2}{n}(-1)^{n+1}2^{2n-1}x^{n-1}$$
$$= \sum_{m=0}^{\infty}\binom{1/2}{m+1}(-1)^m 2^{2m+1}x^m,$$

where, to obtain the last equality, we took $m = n-1$. Expressing the binomial coefficient as a product, we further obtain

$$f(x) = \sum_{m=0}^{\infty}\frac{1}{(m+1)!}\left(\prod_{k=0}^{m}\left(\frac{1}{2}-k\right)\right)(-1)^m 2^{2m+1}x^m$$
$$= \sum_{m=0}^{\infty}\frac{1}{(m+1)!}\left(\prod_{k=1}^{m}\left(k-\frac{1}{2}\right)\right)\frac{1}{2}\cdot 2^{2m+1}x^m = \sum_{m=0}^{\infty}\frac{2^m x^m}{(m+1)!}\prod_{k=1}^{m}(2k-1).$$

Noting that the radius of convergence of this series is indeed positive, as our assumption requires for the power series representing $f(x)$,[2] it follows that

$$B_m = \frac{2^m}{(m+1)!}\prod_{k=1}^{m}(2k-1) \qquad (m \geq 0).$$

10) (SENIOR 6) Suppose that $\{a_n\}_{n=1}^{\infty}$ is a sequence of real numbers such that $\sum_{n=1}^{\infty} a_n b_n$ converges for every sequence $\{b_n\}_{n=1}^{\infty}$ satisfying $\sum_{n=1}^{\infty}|b_n| < +\infty$. Show that the sequence $\{a_n\}_{n=1}^{\infty}$ is bounded.

Source: Problem 6, Real Analysis Comprehensive Exam, August 2011, Oklahoma State University. See

`https://www.math.okstate.edu/node/724`

Solution: Assume that the sequence $\{a_n\}_{n=1}^{\infty}$ satisfies the assumptions given above, and yet it is not bounded. Let f be an increasing function from the set of positive integers into the set of positive integers such that $|a_{f(n)}| > n^2$. Let $b_{f(n)} = 1/n^2$ for all n, and $b_m = 0$ for all positive integers m not in the range of the function f. Then $\sum_{n=1}^{\infty}|b_n| = \sum_{n=1}^{\infty} 1/n^2$ is convergent, and yet $\sum_{n=1}^{\infty} a_n b_n$ is divergent, since the sequence $\{a_n b_n\}_{n=1}^{\infty}$ does not converge to zero. This contradicts our assumptions, establishing the claim in the problem.

11) (SENIOR 7) Let $P(x)$ be a polynomial of degree n for which $P(x) \geq 0$ for all real numbers x. Prove that

$$\sum_{k=0}^{n} P^{(k)}(x) \geq 0$$

for all real numbers x.

[2]The easiest way to establish that this series has a positive radius of convergence is by noting that the radius of convergence of the binomial series is 1, though it is not too difficult to establish this from the last expression obtained for $f(x)$. The radius of convergence of this series is $1/4$. In effect we said this much when we mentioned that the expression obtained for $f(x)$ is valid for $|4x| < 1$.

Source: Problem 10 on the Sixth Annual Iowa Collegiate Mathematics Competition, April 1, 2000. See

`http://sections.maa.org/iowa/Activities/Contest/Problems/Probs00.htm`

for the particular competition, and see

`http://sections.maa.org/iowa/Activities/Contest/`

for the Iowa Collegiate Mathematics Competition.

Solution: Writing

$$Q(x) = \sum_{k=0}^{n} P^{(k)}(x) \geq 0,$$

we have

$$\begin{aligned} Q'(x) - Q(x) &= \sum_{k=0}^{n} P^{(k+1)}(x) - \sum_{k=0}^{n} P^{(k)}(x) \\ &= P^{(n+1)}(x) - P(x) = -P(x). \end{aligned}$$

Hence

$$(e^{-x}Q(x))' = e^{-x}(Q'(x) - Q(x)) = -e^{-x}P(x),$$

and so for any real numbers x and x_0 we have

$$e^{-x}Q(x) = e^{-x_0}Q(x_0) + \int_x^{x_0} e^{-t}P(t)\,dt.$$

Observe that the leading coefficient of $P(x)$ is positive; if it were negative, $P(x)$ would be negative for large positive x. The leading coefficient of $Q(x)$ being the same as that of $P(x)$, it follows that $Q(x_0) > 0$ provided that x_0 is large enough. Now, given x, choose $x_0 > x$ such that $Q(x_0) > 0$; then it follows from the last displayed formula that $Q(x) > 0$, which is what we wanted to show.