Junior Prize Exam
Spring 2014

1) Given a positive integer n, show that $n^3 + 5n$ is divisible by 6.

2) Show that there are no four consecutive integers (i.e., integers of form n, $n+1$, $n+2$, $n+3$ for some n) each of which is a power with an integer exponent > 1 of an integer.

3) Let a, b, c, d, p, and q be positive integers satisfying $ad - bc = 1$ and $a/b > p/q > c/d$. Prove that $q \geq b + d$.[1]

4) Let $n \geq 0$ be an integer. Show that $3^n + 1$ is not divisible by 8.

5) Let n and k be positive integers. Find the number of k element subsets of the set $\{1, 2, \ldots, n\}$ that contain no consecutive integers (two integers are called consecutive if there difference is 1).

6) Given an integer $n > 0$, show that

$$\sum_{k=0}^{n} \binom{2n}{2k} 3^k$$

is divisible by 2^n

7) Assume that for a triangle with angles α, β, and γ, we have

$$\sin\gamma = \cos\alpha + \cos\beta.$$

Show that α or β must be a right angle.

Soon after the exam, solutions will appear on the Web Site
`http://www.sci.brooklyn.cuny.edu/~mate/prize/2014/`

All computer processing for this manuscript was done under Debian Linux. The Perl programming language was instrumental in collating the problems. $\mathcal{AMS}$-TEX was used for typesetting.

[1] By mistake, the our earlier formulation of the problem omitted d from the list of positive integers, and the text said: Let a, b, c, p, and q be positive integers satisfying $ad - bc = 1$ and $a/b > p/q > c/d$. Prove that $q \geq b + d$.